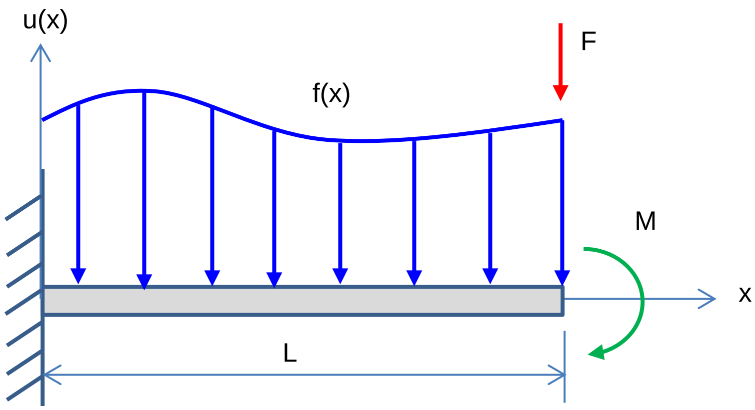
CHAPTER 09: EULER-BERNOULLI'S BEAM THEOREM

EULER-BERNOULLI'S BEAM IN FEM FORM

In this chapter, let's solve Euler-Bernoulli's beam equation by FEM. Let's use the following cantilever problem.



The equation of the above cantilever beam by Euler-Bernoulli's beam theorem (skipping the explanation of this theory) will be **4th order differential equation**.

$$\begin{cases} EI\dfrac{d^4}{dx^4}u(x) - f(x) = 0 \quad \text{.......... Eq.09-1} \\ u(0) = 0 \\ \dfrac{d}{dx}u(0) = 0 \\ EI\dfrac{d^2}{dx^2}u(L) = M \\ -EI\dfrac{d^3}{dx^3}u(L) = F \end{cases}$$

Essential Boundary Conditions

$$u(0) = 0$$

$$\frac{d}{dx}u(0) = 0$$

Natural Boundary Conditions

$$EI\frac{d^2}{dx^2}u(L) = M$$

$$-EI\frac{d^3}{dx^3}u(L) = F$$

where, E: Modulus of Elasticity of the beam

I: Moment of Inertia of the beam

f(x): Distributed load

F: Point load applied at the beam's free edge (x = L)

M: Moment applied at the beams' free edge (x = L)

Using Method of Weighted Residual with a weighing function w(x), we have

$$\int_0^L \left[EI\frac{d^4}{dx^4}u(x) - f(x) \right] w(x)dx = 0$$

Using an integration by parts

$$\Rightarrow \int_0^L \underbrace{EI\frac{d^4}{dx^4}u(x)}_{u'}\underbrace{w(x)}_{v}dx - \int_0^L f(x)w(x)dx = 0$$

Integration by Parts
$\int uv' = uv - \int u'v$
or $\int u'v = uv - \int uv'$

$$\Rightarrow \underbrace{EI\frac{d^3}{dx^3}u(x)}_{u}\underbrace{w(x)}_{v}\Big|_0^L - \int_0^L \underbrace{EI\frac{d^3}{dx^3}u(x)}_{u}\underbrace{\frac{d}{dx}w(x)}_{v'}dx - \int_0^L f(x)w(x)dx = 0$$

$$\Rightarrow \underbrace{EI\frac{d^3}{dx^3}u(L)}_{=-F}w(L) - \underbrace{EI\frac{d^3}{dx^3}u(0)}_{=0}w(0) - \int_0^L EI\frac{d^3}{dx^3}u(x)\frac{d}{dx}w(x)dx - \int_0^L f(x)w(x)dx = 0$$

By boundary condition,
$\frac{d}{dx}u(0) = 0$
Therefore, obviously
$\frac{d^3}{dx^3}u(0) = 0$

$$\Rightarrow -Fw(L) - \int_0^L EI\frac{d^3}{dx^3}u(x)\frac{d}{dx}w(x)dx - \int_0^L f(x)w(x)dx = 0$$

Using an integration by parts again,

$$\Rightarrow -Fw(L) - \int_0^L \underbrace{EI\frac{d^3}{dx^3}u(x)}_{u'}\underbrace{\frac{d}{dx}w(x)}_{v}dx - \int_0^L f(x)w(x)dx = 0$$

Integration by Parts
$\int uv' = uv - \int u'v$
or $\int u'v = uv - \int uv'$

$$\Rightarrow -Fw(L) - \left[\underbrace{EI\frac{d^2}{dx^2}u(x)}_{u}\underbrace{\frac{d}{dx}w(x)}_{v}\Big|_0^L - \int_0^L \underbrace{EI\frac{d^2}{dx^2}u(x)}_{u}\underbrace{\frac{d^2}{dx^2}w(x)}_{v'}dx\right] - \int_0^L f(x)w(x)dx = 0$$

$$\Rightarrow -Fw(L) - \left[\underbrace{EI\frac{d^2}{dx^2}u(L)}_{=M}\frac{d}{dx}w(L) - \underbrace{EI\frac{d^2}{dx^2}u(0)}_{=0}\frac{d}{dx}w(0) - \int_0^L EI\frac{d^2}{dx^2}u(x)\frac{d^2}{dx^2}w(x)dx\right] - \int_0^L f(x)w(x)dx = 0$$

$$\Rightarrow -Fw(L) - \left[M\frac{d}{dx}w(L) - \int_0^L EI\frac{d^2}{dx^2}u(x)\frac{d^2}{dx^2}w(x)dx\right] - \int_0^L f(x)w(x)dx = 0$$

Therefore, we finally obtained

$$\Rightarrow \underbrace{\int_0^L EI\frac{d^2}{dx^2}u(x)\frac{d^2}{dx^2}w(x)dx}_{\text{Stiffness Matrix (a prime shape with a term of w)}} = \int_0^L f(x)w(x)dx + Fw(L) + M\frac{d}{dx}w(L) \quad \text{.. Eq.09-2}$$

Remember in Chapter 04, we talked that the **Galerkin Method** uses N (**shape function**) for w (**weighting function**). Then, what will be the shape function?

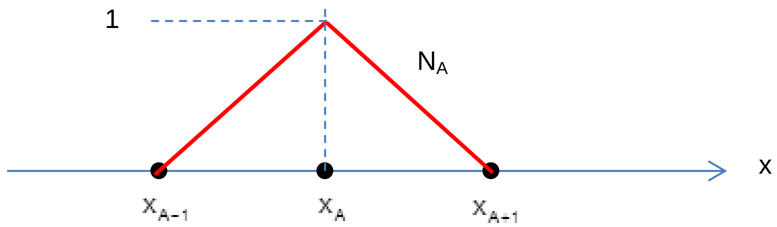
CAN WE USE PIECE-WISE LINEAR SHAPE FUNCTIONS?

So far, we've only used **Piece-wise Linear** shape functions. So, why not use it? Well, let's try.

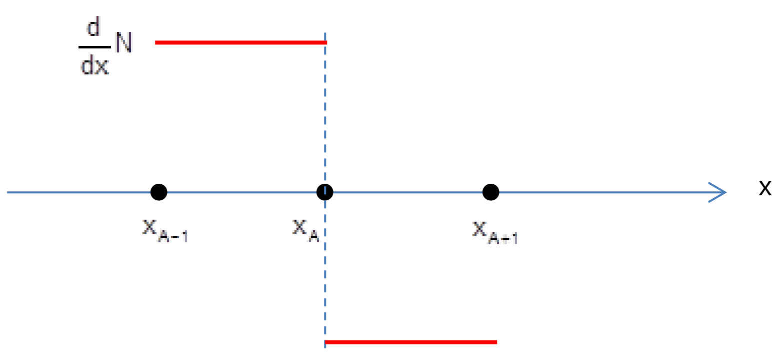
We know the term (stiffness matrix) $\int_0^L EI \frac{d^2}{dx^2}u(x)\frac{d^2}{dx^2}w(x)dx$ has the second derivatives. By Galerkin Method we'll use N for w. From the previous exercise we kind of know that the stiffness matrix will become like $\int_0^L EI \frac{d^2}{dx^2}N_a(x)\frac{d^2}{dx^2}N_b(x)dx$.

Assuming the shape function N is piece-wise linear, then, let's see what happen if we take the second derivative of N.

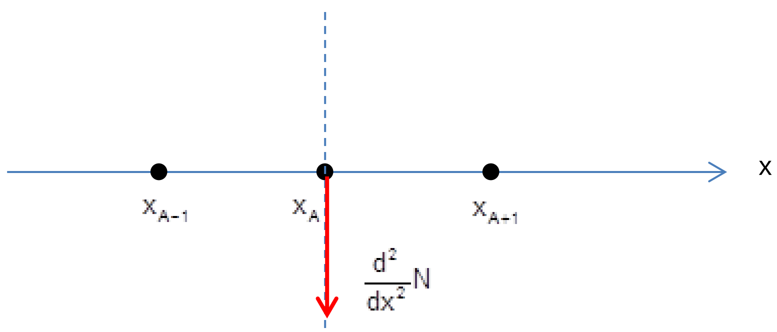
Remember, the piece-wise linear shape function N looks like this.



Taking the first derivative with respect to x once, $\frac{d}{dx}N$ looks



Again, taking the second derivative with respect to x, $\frac{d^2}{dx^2}N$ looks

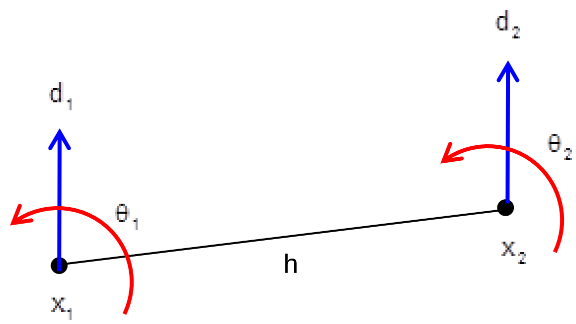


The function is called **Dirac Delta** function and it is not **square integrable**.

Therefore, the term $\int_0^L EI \frac{d^2}{dx^2} N_a(x) \frac{d^2}{dx^2} N_b(x) dx$ cannot be solved if N is piece-wise linear. We cannot use the shape function.

PIECE-WISE CUBIC SHAPE FUNCTION (HERMITE SHAPE FUNCTION)

Let's review an element in Euler-Bernoulli's beam.



d_1 d_2: displacement
θ_1 θ_2: rotation

As shown in the figure above, a node at each end has 2 degrees of freedom: displacement and rotation. Therefore, as one element, there are total 4 degrees of freedoms. The displacement vector has a form as

$$\bar{d} = \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \end{Bmatrix}$$

Therefore, there are 4 unknowns per element (compared to 2 unknowns per element in the previous 2^{nd} order problem). This means we require 4 shape functions per element. So, the question is, what kind of shape function is that?

The answer is called Hermite Shape Function, which has the following form.

$$N_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \qquad \text{(i = 1, 2, 3, and 4)}$$

We'll use element coordinate system for ease of computation: ξ in $[\xi_1, \xi_2] = [-1, 1]$.

Since we need 4 shape functions, we eventually need the following 4 shape functions.

$$N_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3$$

$$N_2 = a_2 + b_2\xi + c_2\xi^2 + d_2\xi^3$$

$$N_3 = a_3 + b_3\xi + c_3\xi^2 + d_3\xi^3$$

$$N_4 = a_4 + b_4\xi + c_4\xi^2 + d_4\xi^3$$

Assuming the approximate solution has the following form,

$$\hat{u} = N_1 d_1 + \frac{d}{dx}u(x)N_3\theta_1 + N_2 d_2 + \frac{d}{dx}u(x)N_4\theta_2$$

Then, each shape function must satisfy the following conditions.

	$N_1(\xi)$	$\frac{d}{d\xi}N_1(\xi)$	$N_2(\xi)$	$\frac{d}{d\xi}N_2(\xi)$	$N_3(\xi)$	$\frac{d}{d\xi}N_3(\xi)$	$N_4(\xi)$	$\frac{d}{d\xi}N_4(\xi)$
ξ_1 (=-1)	1	0	0	0	0	1	0	0
ξ_2 (=+1)	0	0	1	0	0	0	0	1

Therefore, we can identify the unknown coefficients a_i, b_i, c_i, and d_i.

<u>For N_1</u>

$$N_1(\xi_1) = 1$$

$$\Rightarrow a_1 + b_1\xi_1 + c_1(\xi_1)^2 + d_1(\xi_1)^3 = 1$$

$$\Rightarrow a_1 + b_1(-1) + c_1(-1)^2 + d_1(-1)^3 = 1$$

$$\Rightarrow a_1 - b_1 + c_1 - d_1 = 1$$

$$N_1(\xi_2) = 0$$

$$\Rightarrow a_1 + b_1\xi_2 + c_1(\xi_2)^2 + d_1(\xi_2)^3 = 0$$

$$\Rightarrow a_1 + b_1(1) + c_1(1)^2 + d_1(1)^3 = 0$$

$$\Rightarrow a_1 + b_1 + c_1 + d_1 = 0$$

$$\frac{d}{d\xi}N_1(\xi_1) = 0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3\right)\Big|_{\xi=\xi_1} = 0$$

$$\Rightarrow b_1 + 2c_1\xi_1 + 3d_1\xi_1^2 = 0$$

$$\Rightarrow b_1 + 2c_1(-1) + 3d_1(-1)^2 = 0$$

$$\Rightarrow b_1 - 2c_1 + 3d_1 = 0$$

$$\frac{d}{d\xi}N_1(\xi_2) = 0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3\right)\Big|_{\xi=\xi_2} = 0$$

$$\Rightarrow b_1 + 2c_1\xi_2 + 3d_1\xi_2^2 = 0$$

$$\Rightarrow b_1 + 2c_1(1) + 3d_1(1)^2 = 0$$

$$\Rightarrow b_1 + 2c_1 + 3d_1 = 0$$

Therefore, solving for unknown coefficients,

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{Bmatrix} = \begin{Bmatrix} 1/2 \\ -3/4 \\ 0 \\ 1/4 \end{Bmatrix}$$

Therefore, the shape function is,

$$N_1 = a_1 + b_1\xi + c_1\xi^2 + d_1\xi^3 = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3 = \frac{1}{4}(1-\xi)^2(2+\xi) \quad \text{..Eq.09-3}$$

<u>For N_2</u>

$$N_2(\xi_1) = 0$$

$$\Rightarrow a_2 + b_2\xi_1 + c_2(\xi_1)^2 + d_2(\xi_1)^3 = 0$$

$$\Rightarrow a_2 + b_2(-1) + c_2(-1)^2 + d_2(-1)^3 = 0$$

$$\Rightarrow a_2 - b_2 + c_2 - d_2 = 0$$

$$N_2(\xi_2) = 1$$

$$\Rightarrow a_2 + b_2\xi_2 + c_2(\xi_2)^2 + d_2(\xi_2)^3 = 1$$

$$\Rightarrow a_2 + b_2(1) + c_2(1)^2 + d_2(1)^3 = 1$$

$$\Rightarrow a_2 + b_2 + c_2 + d_2 = 1$$

$$\frac{d}{d\xi} N_2(\xi_1) = 0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_2 + b_2\xi + c_2\xi^2 + d_2\xi^3\right)\Big|_{\xi=\xi_1} = 0$$

$$\Rightarrow b_2 + 2c_2\xi_1 + 3d_2\xi_1^2 = 0$$

$$\Rightarrow b_2 + 2c_2(-1) + 3d_2(-1)^2 = 0$$

$$\Rightarrow b_2 - 2c_2 + 3d_2 = 0$$

$$\frac{d}{d\xi}N_2(\xi_2)=0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_2+b_2\xi+c_2\xi^2+d_2\xi^3\right)\Big|_{\xi=\xi_2}=0$$

$$\Rightarrow b_2+2c_2\xi_2+3d_2\xi_2^2=0$$

$$\Rightarrow b_2+2c_2(1)+3d_2(1)^2=0$$

$$\Rightarrow b_2+2c_2+3d_2=0$$

Therefore, solving for unknown coefficients,

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{Bmatrix}=\begin{Bmatrix} 1/2 \\ 3/4 \\ 0 \\ -1/4 \end{Bmatrix}$$

Therefore, the shape function is,

$$N_2=a_2+b_2\xi+c_2\xi^2+d_2\xi^3=\frac{1}{2}+\frac{3}{4}\xi-\frac{1}{4}\xi^3=\frac{1}{4}(1+\xi)^2(2-\xi) \quad \text{...Eq.09-4}$$

<u>For N_3</u>

$$N_3(\xi_1)=0$$

$$\Rightarrow a_3+b_3\xi_1+c_3(\xi_1)^2+d_3(\xi_1)^3=0$$

$$\Rightarrow a_3+b_3(-1)+c_3(-1)^2+d_3(-1)^3=0$$

$$\Rightarrow a_3-b_3+c_3-d_3=0$$

$$N_3(\xi_2)=0$$

$$\Rightarrow a_3+b_3\xi_2+c_3(\xi_2)^2+d_3(\xi_2)^3=0$$

$$\Rightarrow a_3+b_3(1)+c_3(1)^2+d_3(1)^3=0$$

$$\Rightarrow a_3+b_3+c_3+d_3=0$$

$$\frac{d}{d\xi}N_3(\xi_1)=1$$

$$\Rightarrow \frac{d}{d\xi}\left(a_3+b_3\xi+c_3\xi^2+d_3\xi^3\right)\Big|_{\xi=\xi_1}=1$$

$$\Rightarrow b_3 + 2c_3\xi_1 + 3d_3\xi_1^2 = 1$$

$$\Rightarrow b_3 + 2c_3(-1) + 3d_3(-1)^2 = 1$$

$$\Rightarrow b_3 - 2c_3 + 3d_3 = 1$$

$$\frac{d}{d\xi}N_3(\xi_2) = 0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_3 + b_3\xi + c_3\xi^2 + d_3\xi^3\right)\Big|_{\xi=\xi_2} = 0$$

$$\Rightarrow b_3 + 2c_3\xi_2 + 3d_3\xi_2^2 = 0$$

$$\Rightarrow b_3 + 2c_3(1) + 3d_3(1)^2 = 0$$

$$\Rightarrow b_3 + 2c_3 + 3d_3 = 0$$

Therefore, solving for unknown coefficients,

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 1/4 \end{Bmatrix}$$

Therefore, the shape function is,

$$N_3 = a_3 + b_3\xi + c_3\xi^2 + d_3\xi^3 = \frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3 = \frac{1}{4}(1-\xi)^2(1+\xi) \quad \text{................................Eq.09-5}$$

<u>For N_4</u>

$$N_4(\xi_1) = 0$$

$$\Rightarrow a_4 + b_4\xi_1 + c_4(\xi_1)^2 + d_4(\xi_1)^3 = 0$$

$$\Rightarrow a_4 + b_4(-1) + c_4(-1)^2 + d_4(-1)^3 = 0$$

$$\Rightarrow a_4 - b_4 + c_4 - d_4 = 0$$

$$N_4(\xi_2) = 0$$

$$\Rightarrow a_4 + b_4\xi_2 + c_4(\xi_2)^2 + d_4(\xi_2)^3 = 0$$

$$\Rightarrow a_4 + b_4(1) + c_4(1)^2 + d_4(1)^3 = 0$$

$$\Rightarrow a_4 + b_4 + c_4 + d_4 = 0$$

$$\frac{d}{d\xi}N_4(\xi_1)=0$$

$$\Rightarrow \frac{d}{d\xi}\left(a_4+b_4\xi+c_4\xi^2+d_4\xi^3\right)\Big|_{\xi=\xi_1}=0$$

$$\Rightarrow b_4+2c_4\xi_1+3d_4\xi_1^2=0$$

$$\Rightarrow b_4+2c_4(-1)+3d_4(-1)^2=0$$

$$\Rightarrow b_4-2c_4+3d_4=0$$

$$\frac{d}{d\xi}N_4(\xi_2)=1$$

$$\Rightarrow \frac{d}{d\xi}\left(a_4+b_4\xi+c_4\xi^2+d_4\xi^3\right)\Big|_{\xi=\xi_2}=1$$

$$\Rightarrow b_4+2c_4\xi_2+3d_4\xi_2^2=1$$

$$\Rightarrow b_4+2c_4(1)+3d_4(1)^2=1$$

$$\Rightarrow b_4+2c_4+3d_4=1$$

Therefore, solving for unknown coefficients,

$$\begin{bmatrix}1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\\ 0 & 1 & -2 & 3\\ 0 & 1 & 2 & 3\end{bmatrix}\begin{Bmatrix}a_4\\ b_4\\ c_4\\ d_4\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\\ 1\end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix}a_4\\ b_4\\ c_4\\ d_4\end{Bmatrix}=\begin{Bmatrix}-1/4\\ -1/4\\ 1/4\\ 1/4\end{Bmatrix}$$

Therefore, the shape function is,

$$N_4=a_4+b_4\xi+c_4\xi^2+d_4\xi^3=-\frac{1}{4}-\frac{1}{4}\xi+\frac{1}{4}\xi^2+\frac{1}{4}\xi^3=\frac{1}{4}(1+\xi)^2(\xi-1) \quad \text{.............................Eq.09-6}$$

HERMITE SHAPE FUNCTION

These shape functions are called **Hermite Shape Functions**. In summary,

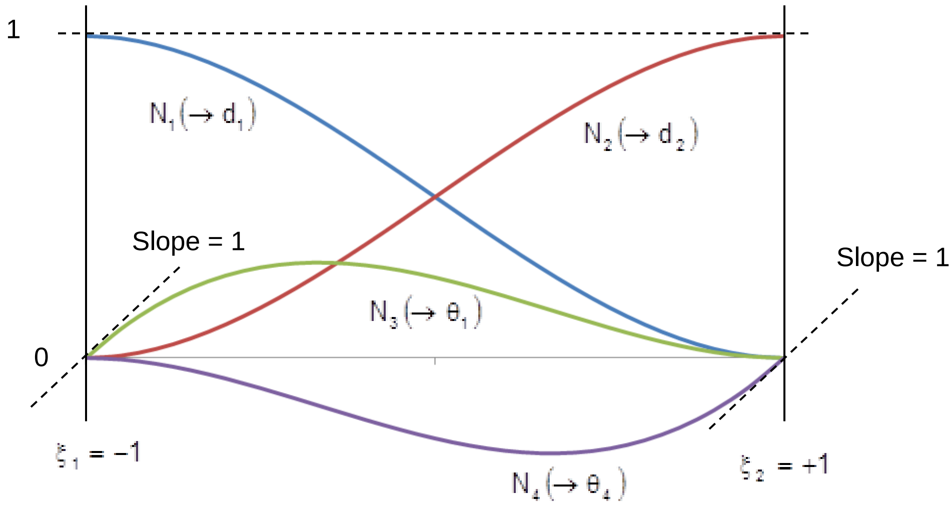
$$N_1=\frac{1}{4}(1-\xi)^2(2+\xi) \qquad \text{(Eq.09-3)}$$

$$N_2=\frac{1}{4}(1+\xi)^2(2-\xi) \qquad \text{(Eq.09-4)}$$

$$N_3=\frac{1}{4}(1-\xi)^2(1+\xi) \qquad \text{(Eq.09-5)}$$

$$N_4 = \frac{1}{4}(1+\xi)^2(\xi - 1) \quad \text{(Eq.09-6)}$$

Figure below shows these shape functions.



Remember the approximate solution is in a form of $\hat{u} = N_1 d_1 + \frac{du}{dx} N_3 \theta_1 + N_2 d_2 + \frac{du}{dx} N_4 \theta_2$.

LOCAL STIFFNESS MATRIX

From Eq. 09-2, we found that the global stiffness matrix is as follows.

$$\tilde{\mathbf{K}} = \int_0^L EI \frac{d^2}{dx^2} u(x) \frac{d^2}{dx^2} w(x) dx$$

Remember,

$$u(x) \approx \hat{u}(x) = \sum d_A N_A(x) \qquad \text{and}$$

$$w(x) \approx \sum d_B N_B(x)$$

Therefore, the stiffness matrix in terms of N will be

$$\tilde{\mathbf{K}} = \int_0^L EI \frac{d^2}{dx^2} N_A(x) \frac{d^2}{dx^2} N_B(x) dx$$

In element space,

$$\tilde{\mathbf{K}}^{\mathbf{e}} = \left[k_{ab}^{e}\right] = \int_{x_1}^{x_2} EI \frac{d^2}{dx^2} N_a(x) \frac{d^2}{dx^2} N_b(x) dx$$

Use the change of variable formula,

$$\Rightarrow \left[k_{ab}^{e}\right] = \int_{-1}^{+1} EI \frac{d^2}{d\xi^2} N_a(x(\xi)) \frac{d^2}{d\xi^2} N_b(x(\xi)) \frac{d}{d\xi} x(\xi) d\xi$$

Then, using the chain rule,

$$\Rightarrow \left[k_{ab}^{e}\right] = \int_{-1}^{+1} EI \left[\frac{\frac{d^2}{d\xi^2} N_a(\xi)}{\frac{d^2}{d\xi^2} x(\xi)}\right] \left[\frac{\frac{d^2}{dx^2} N_b(\xi)}{\frac{d^2}{d\xi^2} x(\xi)}\right] \frac{d}{d\xi} x(\xi) d\xi$$

$$= \int_{-1}^{+1} EI \frac{d^2}{d\xi^2} N_a(\xi) \frac{d^2}{d\xi^2} N_b(\xi) \left[\frac{d}{d\xi} x(\xi)\right]^{-3} d\xi$$

x in terms of ξ is, from interpolation relationship,

$$x - x_1 = \frac{x_2 - x_1}{\xi_2 - \xi_1}(\xi - \xi_1)$$

$$\Rightarrow x - x_1 = \frac{x_2 - x_1}{1-(-1)}(\xi - (-1))$$

$$\Rightarrow x = \frac{x_2 - x_1}{2}\xi + \frac{x_1 + x_2}{2}$$

Therefore,

$$\frac{dx}{d\xi} = \frac{x_2 - x_1}{2} = \frac{h}{2}$$

(where, $h = x_2 - x_1$)

Because $\frac{dx}{d\xi} = \frac{h}{2}$ Eq.09-7

$$\Rightarrow \left[k_{ab}^{e}\right] = \int_{-1}^{+1} EI \frac{d^2}{d\xi^2} N_a(\xi) \frac{d^2}{d\xi^2} N_b(\xi) \left(\frac{2}{h}\right)^3 d\xi$$

$$= \frac{8EI}{h^3} \int_{-1}^{+1} EI \frac{d^2}{d\xi^2} N_a(\xi) \frac{d^2}{d\xi^2} N_b(\xi) d\xi$$

So, what is N_a and N_b? Are they same as what we previously found Hermite shape Functions? Answer is no. Remember, our interpolation function has the following form.

$$\hat{u}(x) = N_1 d_1 + \frac{d}{dx} u(x) N_3 \theta_1 + N_2 d_2 + \frac{d}{dx} u(x) N_4 \theta_2$$

Therefore, we have **N** in the following form.

$$\mathbf{N} = \mathbf{N}(x) = [\mathbf{N_1} \quad \mathbf{N_2} \quad \mathbf{N_3} \quad \mathbf{N_4}] = [N_1(x) \quad N_3(x) \quad N_2(x) \quad N_4(x)]$$

That means,

$$\hat{u}(x) = \underbrace{N_1}_{\mathbf{N_1}} d_1 + \frac{du}{dx} \underbrace{N_3}_{\mathbf{N_2}} \theta_1 + \underbrace{N_2}_{\mathbf{N_3}} d_2 + \frac{du}{dx} \underbrace{N_4}_{\mathbf{N_4}} \theta_2$$

(**N** are in global.)

we use bold N (**N**) in order to distinguish it from the previous individual shape function N.

In element space,

$$\hat{u}(\xi) = N_1 d_1 + \frac{d}{d\xi} u(\xi) N_3 \theta_1 + N_2 d_2 + \frac{d}{d\xi} u(\xi) N_4 \theta_2$$

Because $\frac{dx}{d\xi}=\frac{h}{2}$, or $\frac{du}{d\xi}=\frac{h}{2}\frac{du}{dx}$

$$\hat{u}(\xi)=\underbrace{N_1 d_1}_{\mathbf{N_1}}+\frac{du}{dx}\underbrace{\frac{h}{2}N_3}_{\mathbf{N_2}}\theta_1+\underbrace{N_2 d_2}_{\mathbf{N_3}}+\frac{du}{dx}\underbrace{\frac{h}{2}N_4}_{\mathbf{N_4}}\theta_2 \qquad \text{(}\mathbf{N}\text{ are in local.)}$$

Therefore,

$$\mathbf{N}=\mathbf{N}(\xi)=[\mathbf{N_1}\quad \mathbf{N_2}\quad \mathbf{N_3}\quad \mathbf{N_4}]=\left[N_1(\xi)\quad \frac{h}{2}N_3(\xi)\quad N_2(\xi)\quad \frac{h}{2}N_3(\xi)\right]$$

The second derivative of **N** is,

$$\frac{d^2}{d\xi^2}\mathbf{N}(\xi)=\left[\frac{d^2}{d\xi^2}N_1(\xi)\quad \frac{h}{2}\cdot\frac{d^2}{d\xi^2}N_3(\xi)\quad \frac{d^2}{d\xi^2}N_2(\xi)\quad \frac{h}{2}\cdot\frac{d^2}{d\xi^2}N_4(\xi)\right]$$

Each component of the second derivative of **N** is,

$$\frac{d^2}{d\xi^2}\mathbf{N_1}=\frac{d^2}{d\xi^2}N_1=\frac{d^2}{d\xi^2}\left[\frac{1}{2}-\frac{3}{4}\xi+\frac{1}{4}\xi^3\right]=\frac{3}{2}\xi$$

$$\frac{d^2}{d\xi^2}\mathbf{N_2}=\frac{h}{2}\frac{d^2}{d\xi^2}N_3=\frac{h}{2}\frac{d^2}{d\xi^2}\left[\frac{1}{4}-\frac{1}{4}\xi-\frac{1}{4}\xi^2+\frac{1}{4}\xi^3\right]=\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)$$

$$\frac{d^2}{d\xi^2}\mathbf{N_3}=\frac{d^2}{d\xi^2}N_2=\frac{d^2}{d\xi^2}\left[\frac{1}{2}+\frac{3}{4}\xi-\frac{1}{4}\xi^3\right]=-\frac{3}{2}\xi$$

$$\frac{d^2}{d\xi^2}\mathbf{N_4}=\frac{h}{2}\frac{d^2}{d\xi^2}N_4=\frac{h}{2}\frac{d^2}{d\xi^2}\left[-\frac{1}{4}-\frac{1}{4}\xi+\frac{1}{4}\xi^2+\frac{1}{4}\xi^3\right]=\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)$$

Now, find each component of the stiffness matrix.

$$[k_{11}^e]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_1}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\left(\frac{3}{2}\xi\right)\left(\frac{3}{2}\xi\right)d\xi$$

$$=\frac{18EI}{h^3}\left(\frac{1}{3}\xi^3\right)\Bigg|_{-1}^{+1}=\frac{6EI}{h^3}[1-(-1)]=\frac{12EI}{h^3}$$

$$[k_{12}^e]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_2}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\left(\frac{3}{2}\xi\right)\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$= \frac{3EI}{h^2}\int_{-1}^{+1}\left(3\xi^2 - \xi\right)d\xi$$

$$= \frac{3EI}{h^2}\left(\xi^3 - \frac{1}{2}\xi^2\right)\Bigg|_{-1}^{+1} = \frac{3EI}{h^2}\left[\left(1-\frac{1}{2}\right)-\left(-1-\frac{1}{2}\right)\right] = \frac{6EI}{h^2}$$

$$\left[k_{13}^e\right] = \frac{8EI}{h^3}\int_{-1}^{+1} EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_3}d\xi$$

$$= \frac{8EI}{h^3}\int_{-1}^{+1}\left(\frac{3}{2}\xi\right)\left(-\frac{3}{2}\xi\right)d\xi$$

$$= -\frac{18EI}{h^3}\left(\frac{1}{3}\xi^3\right)\Bigg|_{-1}^{+1} = \frac{6EI}{h^3}\left[1-(-1)\right] = -\frac{12EI}{h^3}$$

$$\left[k_{14}^e\right] = \frac{8EI}{h^3}\int_{-1}^{+1} EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi$$

$$= \frac{8EI}{h^3}\int_{-1}^{+1}\left(\frac{3}{2}\xi\right)\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$= \frac{3EI}{h^2}\int_{-1}^{+1}\left(3\xi^2 + \xi\right)d\xi$$

$$= \frac{3EI}{h^2}\left(\xi^3 + \frac{1}{2}\xi^2\right)\Bigg|_{-1}^{+1} = \frac{3EI}{h^2}\left[\left(1+\frac{1}{2}\right)-\left(-1+\frac{1}{2}\right)\right] = \frac{6EI}{h^2}$$

$$\left[k_{21}^e\right] = \frac{8EI}{h^3}\int_{-1}^{+1} EI\frac{d^2}{d\xi^2}\mathbf{N_2}\frac{d^2}{d\xi^2}\mathbf{N_1}d\xi$$

$$= \frac{8EI}{h^3}\int_{-1}^{+1} EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_2}d\xi = \frac{6EI}{h^2}$$

$$\left[k_{22}^e\right] = \frac{8EI}{h^3}\int_{-1}^{+1} EI\frac{d^2}{d\xi^2}\mathbf{N_2}\frac{d^2}{d\xi^2}\mathbf{N_2}d\xi$$

$$= \frac{8EI}{h^3}\int_{-1}^{+1}\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$= \frac{EI}{2h}\int_{-1}^{+1}(3\xi-1)^2 d\xi$$

$$= \frac{EI}{2h}\left(3\xi^3 - 3\xi^2 + \xi\right)\Big|_{-1}^{+1} = \frac{EI}{2h}\left[(3-3+1)-(-3-3-1)\right] = \frac{4EI}{h}$$

$$\left[k_{23}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_2\frac{d^2}{d\xi^2}\mathbf{N}_3 d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)\left(-\frac{3}{2}\xi\right)d\xi$$

$$=-\frac{3EI}{h^2}\int_{-1}^{+1}\left(3\xi^2-\xi\right)d\xi$$

$$=-\frac{3EI}{h^2}\left(\xi^3-\frac{1}{2}\xi^2\right)\Bigg|_{-1}^{+1}=-\frac{3EI}{h^2}\left[\left(1-\frac{1}{2}\right)-\left(-1-\frac{1}{2}\right)\right]=-\frac{6EI}{h^2}$$

$$\left[k_{24}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_2\frac{d^2}{d\xi^2}\mathbf{N}_4 d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\frac{h}{2}\left(-\frac{1}{2}+\frac{3}{2}\xi\right)\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$=\frac{EI}{2h}\int_{-1}^{+1}\left(-1+3\xi\right)\left(1+3\xi\right)d\xi$$

$$=-\frac{EI}{2h}\int_{-1}^{+1}\left(9\xi^2-1\right)d\xi$$

$$=-\frac{EI}{2h}\left(\frac{1}{3}\xi^3-\xi\right)\Bigg|_{-1}^{+1}=-\frac{3EI}{2h}\left[\left(\frac{1}{3}-1\right)-\left(-\frac{1}{3}+1\right)\right]=\frac{2EI}{h}$$

$$\left[k_{31}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_3\frac{d^2}{d\xi^2}\mathbf{N}_1 d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_1\frac{d^2}{d\xi^2}\mathbf{N}_3=-\frac{12EI}{h^3}$$

$$\left[k_{32}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_3\frac{d^2}{d\xi^2}\mathbf{N}_2 d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_2\frac{d^2}{d\xi^2}\mathbf{N}_3=-\frac{6EI}{h^2}$$

$$\left[k_{33}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N}_3\frac{d^2}{d\xi^2}\mathbf{N}_3 d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\left(-\frac{3}{2}\xi\right)\left(-\frac{3}{2}\xi\right)d\xi$$

$$=\frac{18EI}{h^3}\left(\frac{1}{3}\xi^3\right)\Bigg|_{-1}^{+1}=\frac{6EI}{h^3}\left[1-\left(-1\right)\right]=\frac{12EI}{h^3}$$

$$\left[k_{34}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_3}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\left(-\frac{3}{2}\xi\right)\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$=\frac{EI}{h^2}\int_{-1}^{+1}(-3\xi)(1+3\xi)d\xi$$

$$=-\frac{EI}{h^2}\int_{-1}^{+1}\left(9\xi^2+3\xi\right)d\xi$$

$$=-\frac{EI}{h^2}\left(3\xi^3+\frac{3}{2}\xi^2\right)\Bigg|_{-1}^{+1}=-\frac{EI}{h^2}\left[\left(3+\frac{3}{2}\right)-\left(-3+\frac{3}{2}\right)\right]=-\frac{6EI}{h^2}$$

$$\left[k_{41}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_4}\frac{d^2}{d\xi^2}\mathbf{N_1}d\xi$$

$$\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_1}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi=\frac{6EI}{h^2}$$

$$\left[k_{42}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_4}\frac{d^2}{d\xi^2}\mathbf{N_2}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_2}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi=\frac{2EI}{h}$$

$$\left[k_{43}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_4}\frac{d^2}{d\xi^2}\mathbf{N_3}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_3}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi=-\frac{6EI}{h^2}$$

$$\left[k_{44}^{e}\right]=\frac{8EI}{h^3}\int_{-1}^{+1}EI\frac{d^2}{d\xi^2}\mathbf{N_4}\frac{d^2}{d\xi^2}\mathbf{N_4}d\xi$$

$$=\frac{8EI}{h^3}\int_{-1}^{+1}\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)\frac{h}{2}\left(\frac{1}{2}+\frac{3}{2}\xi\right)d\xi$$

$$=\frac{EI}{2h}\int_{-1}^{+1}(3\xi+1)^2d\xi$$

$$=\frac{EI}{2h}\left(3\xi^3+3\xi^2+\xi\right)\Big|_{-1}^{+1}=\frac{EI}{2h}\left[(3+3+1)-(-3+3-1)\right]=\frac{4EI}{h}$$

Therefore,

$$[k^e] = \begin{bmatrix} k^e_{11} & k^e_{12} & k^e_{13} & k^e_{14} \\ k^e_{21} & k^e_{22} & k^e_{23} & k^e_{24} \\ k^e_{31} & k^e_{32} & k^e_{33} & k^e_{34} \\ k^e_{41} & k^e_{42} & k^e_{43} & k^e_{44} \end{bmatrix} = \begin{bmatrix} \frac{12EI}{h^3} & \frac{6EI}{h^2} & -\frac{12EI}{h^3} & \frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{4EI}{h} & -\frac{6EI}{h^2} & \frac{2EI}{h} \\ -\frac{12EI}{h^3} & -\frac{6EI}{h^2} & \frac{12EI}{h^3} & -\frac{6EI}{h^2} \\ \frac{6EI}{h^2} & \frac{2EI}{h} & -\frac{6EI}{h^2} & \frac{4EI}{h} \end{bmatrix}$$

Or, in a more common form,

$$[k^e] = \frac{EI}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \quad \text{......... Eq. 09-8}$$

LOCAL FORCE VECTOR

From Eq. 09-2, the RHS will be a force vector.

$$\bar{\mathbf{f}} = \int_0^L f(x)w(x)dx + Fw(L) + M\frac{d}{dx}w(L)$$

Again, using the weighing function as $w(x) \approx \sum d_B N_B(x)$, it can be written as

$$\bar{\mathbf{f}} = \int_0^L f(x)N_B(x)dx + FN_B(L) + M\frac{d}{dx}N_B(x)\Big|_{x=L} \qquad \text{(Note } \sum d_B \text{ term factored out to stiffness matrix.)}$$

Note that terms $FN_B(L)$ and $M\frac{d}{dx}N_B(x)\Big|_{x=L}$ are boundary condition. To develop general force vector, let's forget these terms. So, now we have,

$$\bar{\mathbf{f}} = \int_0^L f(x)N_B(x)dx$$

In element space,

$$\{f^e_b\} = \int_{x_1}^{x_2} f(x)N_b(x)dx$$

Using the change of variable formula,

$$\left\{f_b^e\right\} = \int_{-1}^{+1} f(x(\xi)) N_b(x(\xi)) \frac{d}{d\xi} x(\xi) d\xi$$

Because $\frac{dx}{d\xi} = \frac{h}{2}$ (Eq.09-7)

$$\left\{f_b^e\right\} = \frac{h}{2}\int_{-1}^{+1} f(\xi) N_B(\xi) d\xi$$

In the same way we did for the stiffness matrix, we define **N** for N_B as

$$\mathbf{N} = \mathbf{N}(\xi) = [\mathbf{N}_1 \quad \mathbf{N}_2 \quad \mathbf{N}_3 \quad \mathbf{N}_4] = \left[N_1(\xi) \quad \frac{h}{2}N_3(\xi) \quad N_2(\xi) \quad \frac{h}{2}N_3(\xi) \right]$$

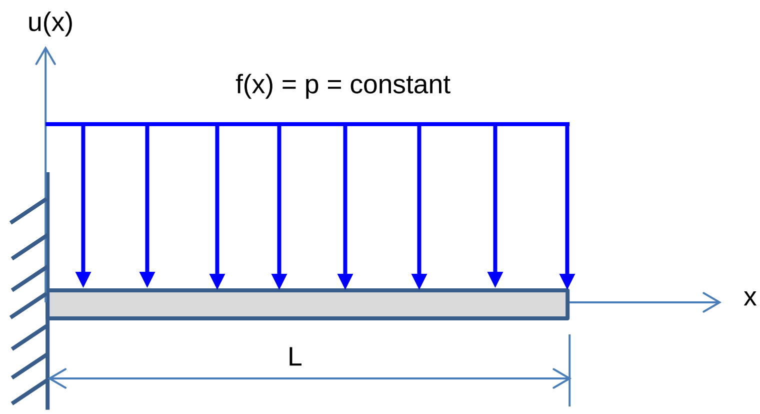
Since the force vector contains a function $f(\xi)$, we cannot simplify it further. Let's do some examples.

EXAMPLE (SIMPLE GENERAL CASE WITH CONSTANT P)

Let's consider the following very simple example. We choose f(x), F, M as follows:

$$\begin{cases} f(x) = p \text{ (therefore, } f(\xi) = p) & \text{where, p is constant} \\ F = 0 \\ M = 0 \end{cases}$$

Therefore, the Euler-Bernoulli's beam problem now looks like this.



Each component of force vector is,

$$\{f_1\} = \frac{h}{2}\int_{-1}^{+1} f(\xi) N_1(\xi) d\xi$$

$$= \frac{h}{2}\int_{-1}^{+1} p\left(\frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^3\right) d\xi$$

See Eq. 09-3 to Eq. 09-6 for N_1 through N_4

$$= \frac{ph}{2}\left(\frac{1}{2}\xi - \frac{3}{8}\xi^2 + \frac{1}{16}\xi^4\right)\Bigg|_{-1}^{+1}$$

$$= \frac{ph}{2}\left[\left(\frac{1}{2} - \frac{3}{8} + \frac{1}{16}\right) - \left(-\frac{1}{2} - \frac{3}{8} + \frac{1}{16}\right)\right] = \frac{ph}{2}$$

$$\{f_2\} = \frac{h}{2}\int_{-1}^{+1} f(\xi) N_2(\xi) d\xi$$

$$= \frac{h}{2}\int_{-1}^{+1} p\frac{h}{2}\left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3\right) d\xi$$

$$= \frac{ph^2}{4}\left(\frac{1}{4}\xi - \frac{1}{8}\xi^2 - \frac{1}{12}\xi^3 + \frac{1}{16}\xi^4\right)\Bigg|_{-1}^{+1}$$

$$= \frac{ph^2}{4}\left[\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{12} + \frac{1}{16}\right) - \left(-\frac{1}{4} - \frac{1}{8} + \frac{1}{12} + \frac{1}{16}\right)\right] = \frac{ph^2}{12}$$

$$\{f_3\} = \frac{h}{2}\int_{-1}^{+1} f(\xi) N_3(\xi) d\xi$$

$$= \frac{h}{2}\int_{-1}^{+1} p\left(\frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^3\right) d\xi$$

$$= \frac{ph}{2}\left(\frac{1}{2}\xi + \frac{3}{8}\xi^2 - \frac{1}{16}\xi^4\right)\Bigg|_{-1}^{+1}$$

$$= \frac{ph}{2}\left[\left(\frac{1}{2} + \frac{3}{8} - \frac{1}{16}\right) - \left(-\frac{1}{2} + \frac{3}{8} - \frac{1}{16}\right)\right] = \frac{ph}{2}$$

$$\{f_4\} = \frac{h}{2}\int_{-1}^{+1} f(\xi) N_4(\xi) d\xi$$

$$= \frac{h}{2}\int_{-1}^{+1} p\frac{h}{2}\left(-\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\xi^2 + \frac{1}{4}\xi^3\right) d\xi$$

$$= \frac{ph^2}{4}\left(-\frac{1}{4}\xi - \frac{1}{8}\xi^2 + \frac{1}{12}\xi^3 + \frac{1}{16}\xi^4\right)\Bigg|_{-1}^{+1}$$

$$= \frac{ph^2}{4}\left[\left(-\frac{1}{4} - \frac{1}{8} + \frac{1}{12} + \frac{1}{16}\right) - \left(\frac{1}{4} - \frac{1}{8} - \frac{1}{12} + \frac{1}{16}\right)\right] = -\frac{ph^2}{12}$$

Therefore,

$$\{f_b^e\} = \begin{Bmatrix} f_1^e \\ f_2^e \\ f_3^e \\ f_4^e \end{Bmatrix} = \begin{Bmatrix} \frac{ph}{2} \\ \frac{ph^2}{12} \\ \frac{ph}{2} \\ -\frac{ph^2}{12} \end{Bmatrix} \quad \text{.......... Eq. 09-9}$$

Therefore, using Eq. 09-8 and Eq. 09-9, the matrix form is,

$$\{f_a^e\} = [k_{ab}]\{d_b\}$$

Or,

$$\begin{Bmatrix} \frac{ph}{2} \\ \frac{ph^2}{12} \\ \frac{ph}{2} \\ -\frac{ph^2}{12} \end{Bmatrix} = \frac{EI}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \end{Bmatrix} \quad \text{.......... Eq.09-10}$$

The value p, h, E, and I are per element. The matrix form is still valid if these values vary for different elements

The global matrix form can be formed from element matrix form as

$$\{F_A\} = [K_{AB}]\{d_B\}$$

where, $\{F_A\} = \sum_{e=1}^{n_{el}} \{f_a^e\}$

$$[K_{AB}] = \sum_{e=1}^{n_{el}} [k_{ab}^e]$$

The assembling global matrix is described in the figure below.

$$
[K_{AB}] = \begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} & K_{18} & \cdots & K_{1n} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} & K_{28} & \cdots & K_{2n} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} & K_{38} & \cdots & K_{3n} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & \cdots & K_{4n} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & \cdots & K_{5n} \\
K_{64} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} & \cdots & K_{6n} \\
K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & \cdots & K_{7n} \\
K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & \cdots & K_{8n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
K_{n1} & K_{n2} & K_{n3} & K_{n4} & K_{n5} & K_{n6} & K_{n7} & K_{n8} & \cdots & K_{nn}
\end{bmatrix}
$$

n x n Matrix

$(n = 2n_{el} + 2)$

$$
\begin{bmatrix}
k^e_{11} & k^e_{12} & k^e_{13} & k^e_{14} \\
k^e_{21} & k^e_{22} & k^e_{23} & k^e_{24} \\
k^e_{31} & k^e_{32} & k^e_{33} & k^e_{34} \\
k^e_{41} & k^e_{42} & k^e_{43} & k^e_{44}
\end{bmatrix}
$$

$$
\{F_A\} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ \vdots \\ f_n \end{Bmatrix}
\qquad
\begin{Bmatrix} f^e_1 \\ f^e_2 \\ f^e_3 \\ f^e_4 \end{Bmatrix}
$$

n x 1 Vector

$(n = 2n_{el} + 2)$

EXAMPLE 1 (CONSTANT E, I, and P)

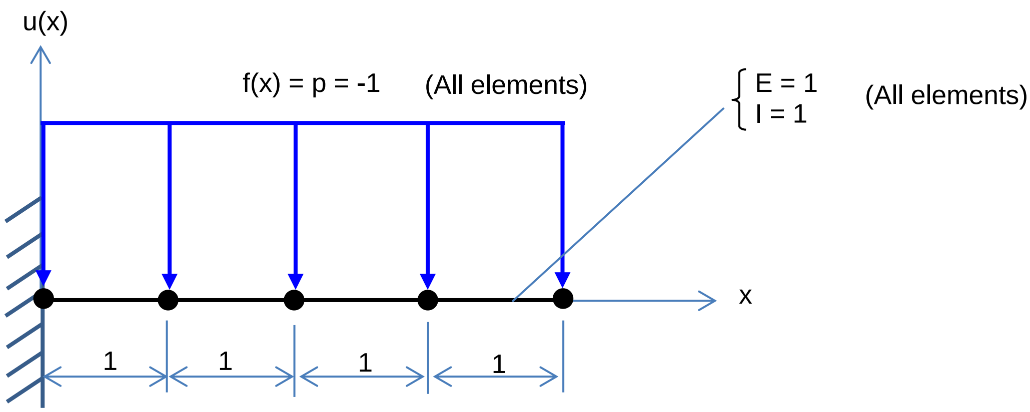
As the first example, let's consider all values are constant, that is,

$$\begin{cases} E_1 = E_2 = E_3 = E_4 = 1 \\ I_1 = I_2 = I_3 = I_4 = 1 \\ p_1 = p_2 = p_3 = p_4 = -1 \end{cases}$$

Let's consider 5 nodes, 4 element problem with each size equal to one, that is,

$$h_1 = h_2 = h_3 = h_4 = 1 \qquad \text{(Therefore, L = 4)}$$

The figure below summarizes the problem.



Exact Solution

Because of constant properties throughout the beam, we have exact solution for this problem.

$$u(x) = \frac{px^2}{24EI}\left(x^2 - 4Lx + 6L^2\right)$$

Refer to any textbooks of mechanics of materials

The deflection at the free-end of the beam (x = L = 4) is,

$$u(4) = \frac{(-1)4^2}{24(1)(1)}\left(4^2 - 4(4)(4) + 6(4)^2\right) = -32$$

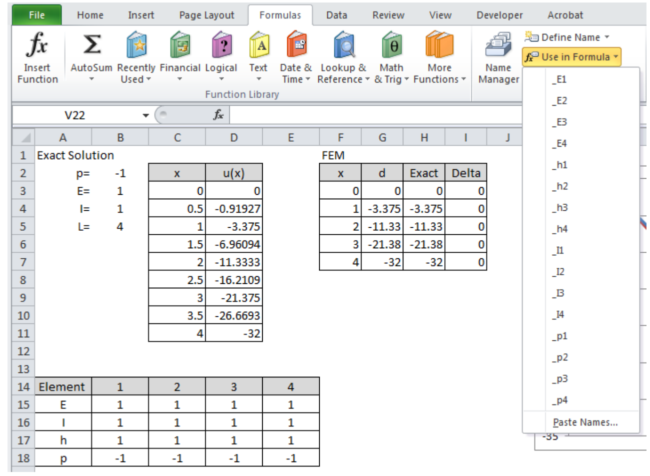
FEM Solution

We use excel to solve the matrix equation.

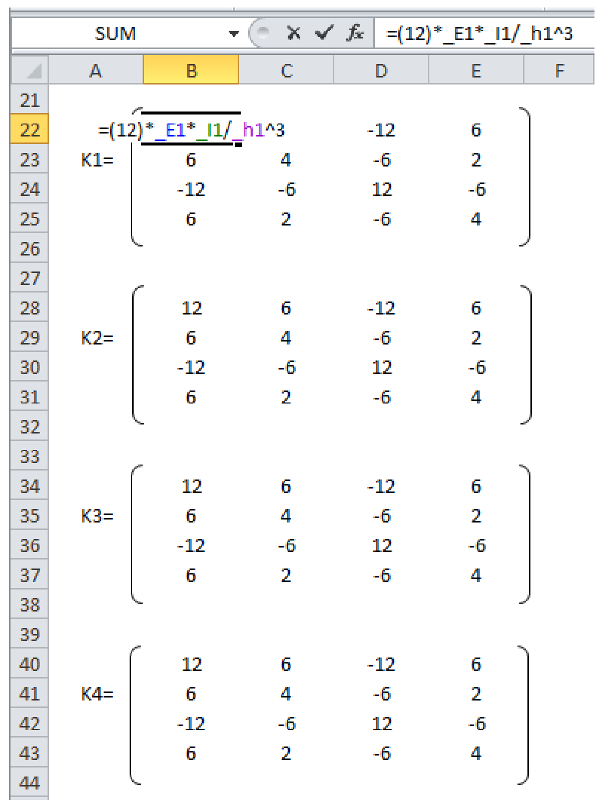
First, let's setup the cells for each property for each element.

	A	B	C	D	E
14	Element	1	2	3	4
15	E	1	1	1	1
16	I	1	1	1	1
17	h	1	1	1	1
18	p	-1	-1	-1	-1

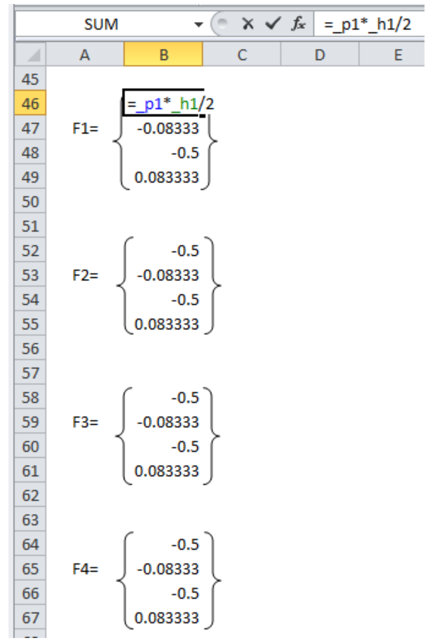
It may be easier if you define names for the above variables.



Using Eq. 09-8, compute stiffness matrix of each element.



Using Eq. 09-9, compute force vector of each element.



Now, assemble a global matrix from each local element stiffness matrix.

Be careful that you have 4 cells overlapping in one local matrix with another local matrix.

$$
K\_G=\begin{pmatrix}
12 & 6 & -12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 4 & -6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 & 0 & 0 \\
6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 \\
0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 \\
0 & 0 & 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4
\end{pmatrix}
$$

Same thing for global force vector.

$$
F\_G=\begin{Bmatrix}
-0.5 \\ -0.083 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -0.5 \\ 0.0833
\end{Bmatrix}
$$

Now, we have the global matrix equation like this.

$$\begin{Bmatrix} -0.5 \\ -0.083 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -0.5 \\ 0.083 \end{Bmatrix} = \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \\ d_3 \\ \theta_3 \\ d_4 \\ \theta_4 \\ d_5 \\ \theta_5 \end{Bmatrix}$$

Remember the under-constraint problem talked in Chapter 07, we cannot solve this with 10 unknowns. However, from the essential boundary conditions, we know

$$d_1 = 0$$

$$\theta_1 = 0$$

Therefore, we are only using the following portion of the global matrix form.

$d_1 = 0$

$\theta_1 = 0$

$$\begin{Bmatrix} -0.5 \\ -0.083 \\ -1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ -0.5 \\ 0.083 \end{Bmatrix} = \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & -6 & 24 & 0 & -12 & 6 \\ 0 & 0 & 0 & 0 & 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \\ d_3 \\ \theta_3 \\ d_4 \\ \theta_4 \\ d_5 \\ \theta_5 \end{Bmatrix}$$

Then, Inverse of K (only highlighted portion of original K) can be computed as:

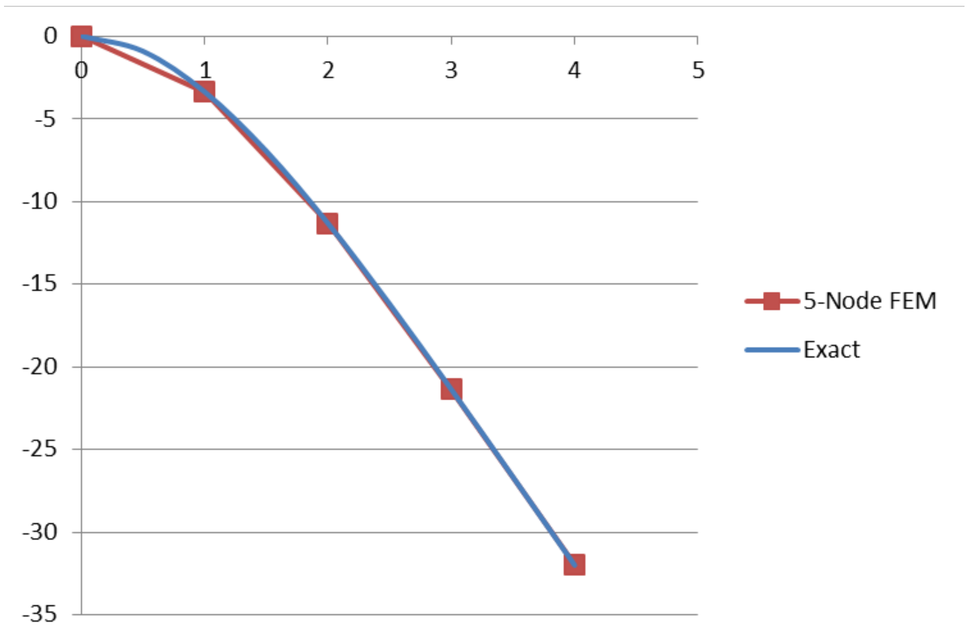
$$KI = \begin{pmatrix} 0.3333 & 0.5 & 0.8333 & 0.5 & 1.3333 & 0.5 & 1.8333 & 0.5 \\ 0.5 & 1 & 1.5 & 1 & 2.5 & 1 & 3.5 & 1 \\ 0.8333 & 1.5 & 2.6667 & 2 & 4.6667 & 2 & 6.6667 & 2 \\ 0.5 & 1 & 2 & 2 & 4 & 2 & 6 & 2 \\ 1.3333 & 2.5 & 4.6667 & 4 & 9 & 4.5 & 13.5 & 4.5 \\ 0.5 & 1 & 2 & 2 & 4.5 & 3 & 7.5 & 3 \\ 1.8333 & 3.5 & 6.6667 & 6 & 13.5 & 7.5 & 21.333 & 8 \\ 0.5 & 1 & 2 & 2 & 4.5 & 3 & 8 & 4 \end{pmatrix}$$

Therefore, displacement vector is,

$$\{d\} = [K]^{-1}\{F\}$$

$$\begin{Bmatrix} d_2 \\ \theta_2 \\ d_3 \\ \theta_3 \\ d_4 \\ \theta_4 \\ d_5 \\ \theta_5 \end{Bmatrix} = \begin{Bmatrix} -3.375 \\ -6.167 \\ -11.33 \\ -9.333 \\ -21.38 \\ -10.5 \\ -32 \\ -10.67 \end{Bmatrix}$$

Below plot compares FEM results and exact solution.



The entire workbook of this problem like below. You can download Excel file from the following link.

http:www.3dwhiffletree.com/FEM/download/Chapter09-1.xlsx

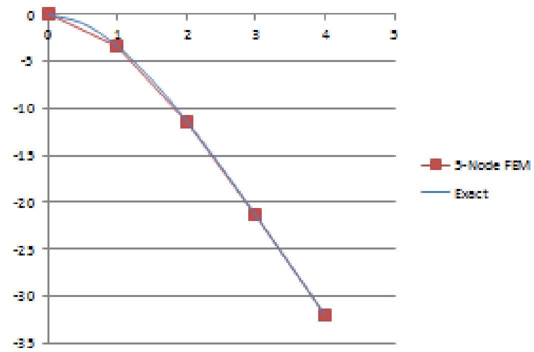
Exact Solution

p=	-1
E=	1
I=	1
L=	4

x	u(x)
0	0
0.5	-0.9193
1	-3.375
1.5	-6.9609
2	-11.333
2.5	-16.211
3	-21.375
3.5	-26.669
4	-32

FEM

x	d	Exact	Delta
0	0	0	0
1	-3.38	-3.38	0
2	-11.3	-11.3	0
3	-21.4	-21.4	0
4	-32	-32	0



Element	1	2	3	4
E	1	1	1	1
I	1	1	1	1
h	1	1	1	1
p	-1	-1	-1	-1

$$K1=\begin{bmatrix}12&6&-12&6\\6&4&-6&2\\-12&-6&12&-6\\6&2&-6&4\end{bmatrix}$$

$$K2=\begin{bmatrix}12&6&-12&6\\6&4&-6&2\\-12&-6&12&-6\\6&2&-6&4\end{bmatrix}$$

$$K3=\begin{bmatrix}12&6&-12&6\\6&4&-6&2\\-12&-6&12&-6\\6&2&-6&4\end{bmatrix}$$

$$K4=\begin{bmatrix}12&6&-12&6\\6&4&-6&2\\-12&-6&12&-6\\6&2&-6&4\end{bmatrix}$$

$$F1=\begin{Bmatrix}-0.5\\-0.0833\\-0.5\\0.0833\end{Bmatrix}\quad F2=\begin{Bmatrix}-0.5\\-0.0833\\-0.5\\0.0833\end{Bmatrix}\quad F3=\begin{Bmatrix}-0.5\\-0.0833\\-0.5\\0.0833\end{Bmatrix}\quad F4=\begin{Bmatrix}-0.5\\-0.0833\\-0.5\\0.0833\end{Bmatrix}$$

$$K\_G=\begin{bmatrix}12&6&-12&6&0&0&0&0&0&0\\6&4&-6&2&0&0&0&0&0&0\\-12&-6&24&0&-12&6&0&0&0&0\\6&2&0&8&-6&2&0&0&0&0\\0&0&-12&-6&24&0&-12&6&0&0\\0&0&6&2&0&8&-6&2&0&0\\0&0&0&0&-12&-6&24&0&-12&6\\0&0&0&0&6&2&0&8&-6&2\\0&0&0&0&0&0&-12&-6&12&-6\\0&0&0&0&0&0&6&2&-6&4\end{bmatrix}\quad \begin{array}{l}U\\d1=0\\the1=0\\d2\\the2\\d3\\the3\\d4\\the4\\d5\\the5\end{array}$$

$$F\_G=\begin{Bmatrix}-0.5\\-0.08\\-1\\0\\-1\\0\\-1\\0\\-0.5\\0.08\end{Bmatrix}$$

$$KI=\begin{bmatrix}0.33&0.5&0.83&0.5&1.33&0.5&1.83&0.5\\0.5&1&1.5&1&2.5&1&3.5&1\\0.83&1.5&2.67&2&4.67&2&6.67&2\\0.5&1&2&2&4&2&6&2\\1.33&2.5&4.67&4&9&4.5&13.5&4.5\\0.5&1&2&2&4.5&3&7.5&3\\1.83&3.5&6.67&6&13.5&7.5&21.3&8\\0.5&1&2&2&4.5&3&8&4\end{bmatrix}$$

$$d=\begin{Bmatrix}-3.38\\-6.17\\-11.3\\-9.33\\-21.4\\-10.5\\-32\\-10.7\end{Bmatrix}\quad \begin{array}{l}d2\\the2\\d3\\the3\\d4\\the4\\d5\\the5\end{array}$$

Node	X	d
1	0	0
2	1	-3.38
3	2	-11.3
4	3	-21.4
5	4	-32

EXAMPLE 2 (VARIABLE h)

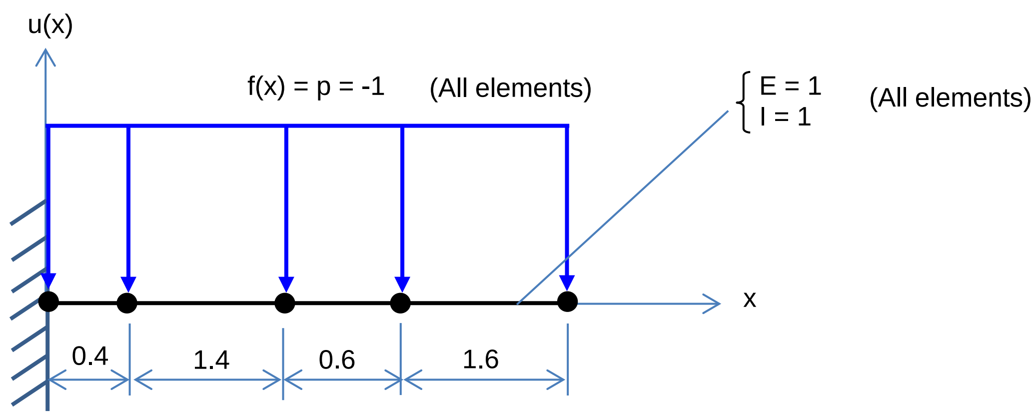
This time, using the same example problem but let's use different mesh sizes. Other properties remain the same as:

$$\begin{cases} E_1 = E_2 = E_3 = E_4 = 1 \\ I_1 = I_2 = I_3 = I_4 = 1 \\ p_1 = p_2 = p_3 = p_4 = 1- \end{cases}$$

New meshes are

$$\begin{cases} h_1 = 0.4 \\ h_2 = 1.4 \\ h_3 = 0.6 \\ h_4 = 1.6 \qquad \text{(Therefore, L = 4)} \end{cases}$$

The figure below summarizes the problem.



Exact Solution

We still can find exact solution for this problem because all other properties are constant throughout the beam. Total L has not changed, therefore, the deflection at the free-end of the beam did not change.

$$u(4) = \frac{(-1)4^2}{24(1)(1)}\left(4^2 - 4(4)(4) + 6(4)^2\right) = -32$$

FEM Solution

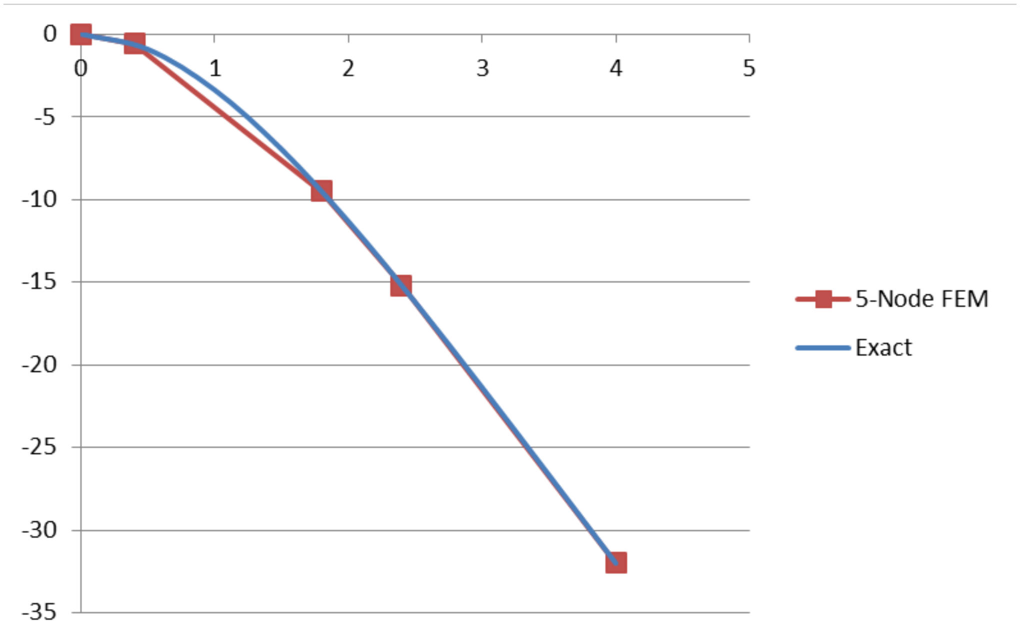
The last Excel spreadsheet was formulated so that it works with variable h. We'll enter new h values as follows.

Element	1	2	3	4
E	1	1	1	1
I	1	1	1	1
h	0.4	1.4	0.6	1.6
p	-1	-1	-1	-1

The Excel spreadsheet will automatically build a new stiffness matrix and give you this result.

$$d=\left\{\begin{matrix} -0.598 \\ -2.891 \\ -9.509 \\ -8.892 \\ -15.21 \\ -9.984 \\ -32 \\ -10.67 \end{matrix}\right\}\begin{matrix} \leftarrow d_2 \\ \\ \leftarrow d_3 \\ \\ \leftarrow d_4 \\ \\ \leftarrow d_5 \\ \\ \end{matrix}$$

Results are plotted below.



You can download Excel file from the following link.

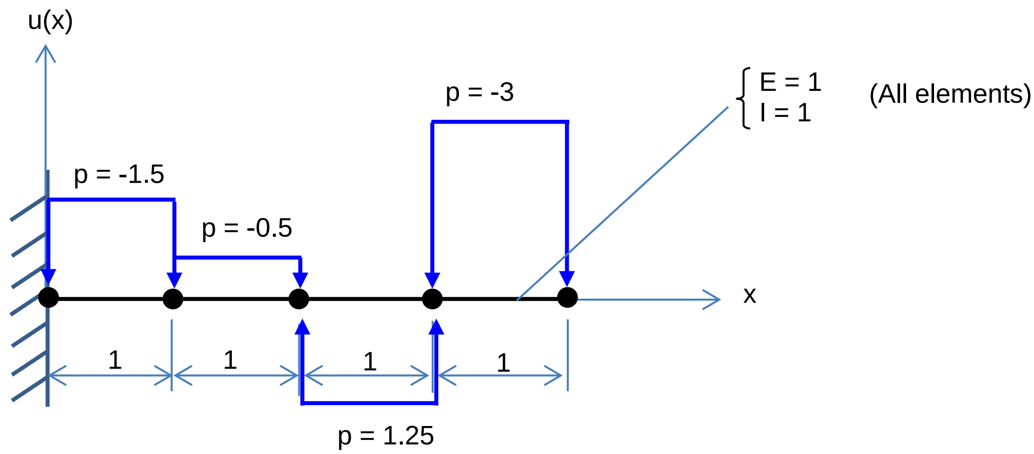
http:www.3dwhiffletree.com/FEM/download/Chapter09-2.xlsx

EXAMPLE 3 (VARIABLE p)

Now, we change load p this time. Everything else is same as Example 1. Therefore,

$$\begin{cases} E_1 = E_2 = E_3 = E_4 = 1 \\ I_1 = I_2 = I_3 = I_4 = 1 \\ p_1 = -1.5 \\ p_2 = -0.5 \\ p_3 = 1.25 \\ p_4 = -3 \\ h_1 = h_2 = h_3 = h_4 = 1 \qquad \text{(Therefore, L = 4)} \end{cases}$$

The figure below summarizes the problem.



FEM Solution

Enter p values for each element.

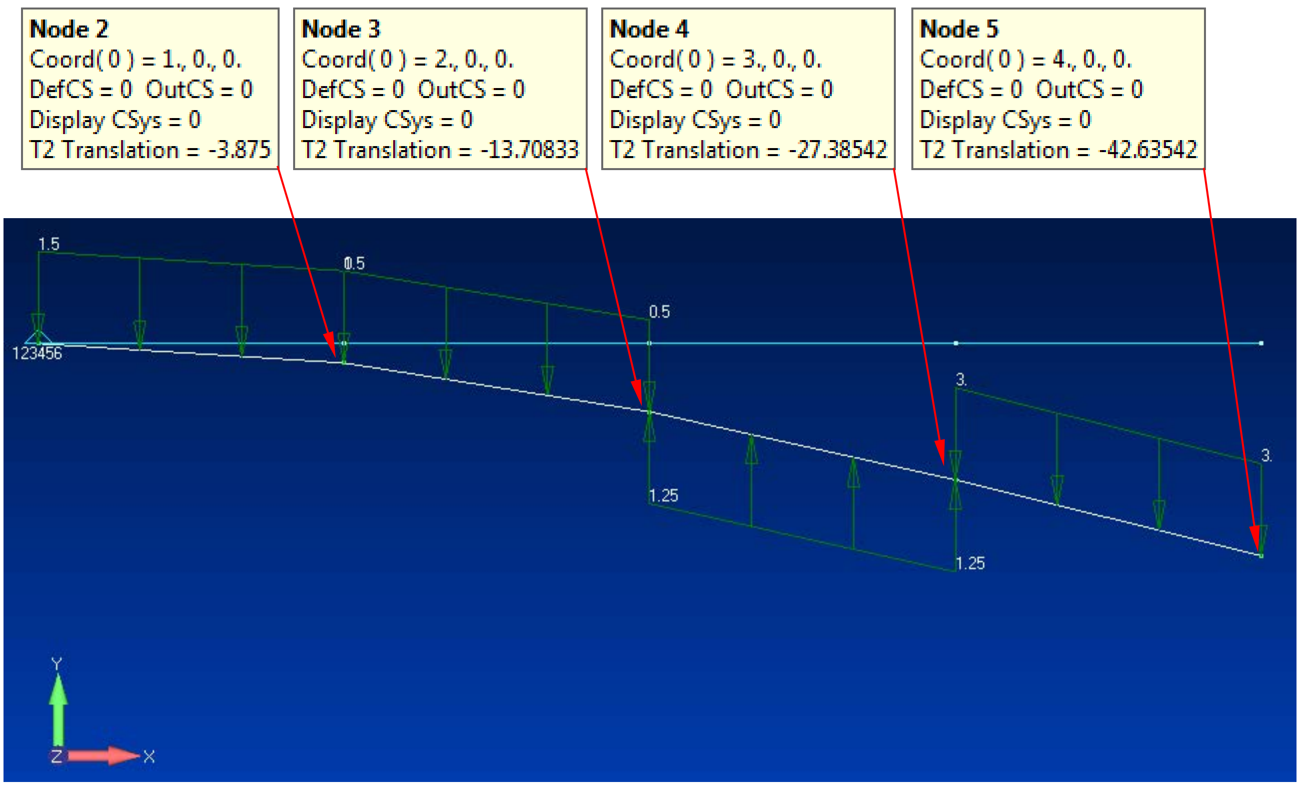
Element	1	2	3	4
E	1	1	1	1
I	1	1	1	1
h	1	1	1	1
p	-1.5	-0.5	1.25	-3

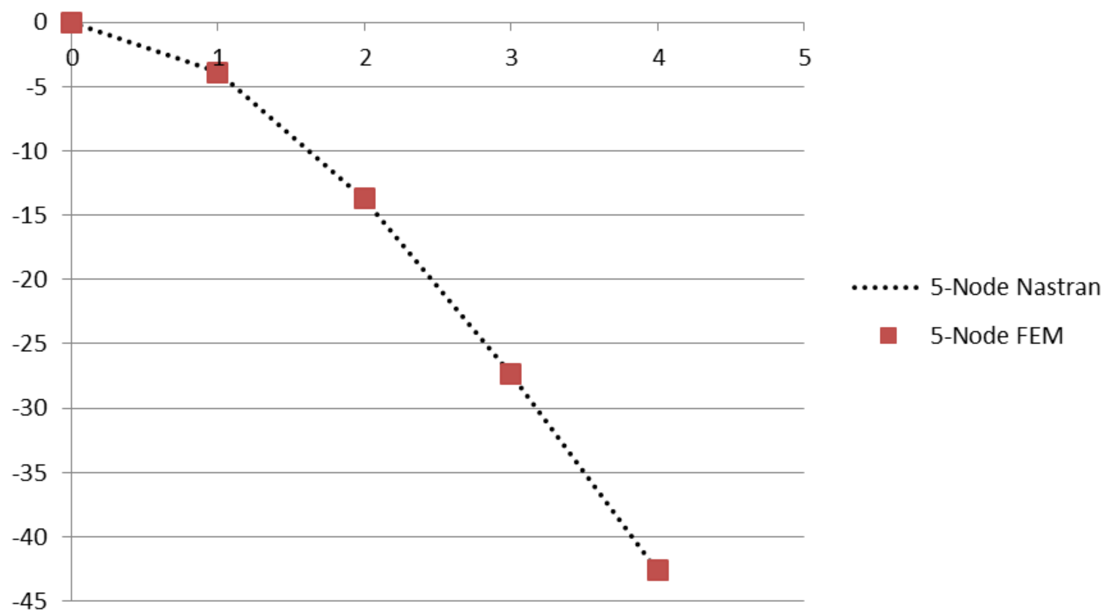
The Excel spreadsheet will automatically build a new stiffness matrix and give you this result.

$$d=\begin{array}{c l}\left\{\begin{matrix} -3.875 \\ -7.25 \\ -13.71 \\ -12.08 \\ -27.39 \\ -14.88 \\ -42.64 \\ -15.38 \end{matrix}\right\} & \begin{matrix} \leftarrow d_2 \\ \\ \leftarrow d_3 \\ \\ \leftarrow d_4 \\ \\ \leftarrow d_5 \\ \\ \end{matrix}\end{array}$$

Nastran Solution

Since we cannot find an exact solution this time (it's not impossible to solve the equation by hand though), this time I want to compare our FEM results with Nastran solution. Below shows the results.





You can download both Excel file and Nastran files from the following links.

Excel File:
http:www.3dwhiffletree.com/FEM/download/Chapter09-3.xlsx

FEMAP neutral file (if you have FEMAP)
http:www.3dwhiffletree.com/FEM/download/Chapter09-3.NEU

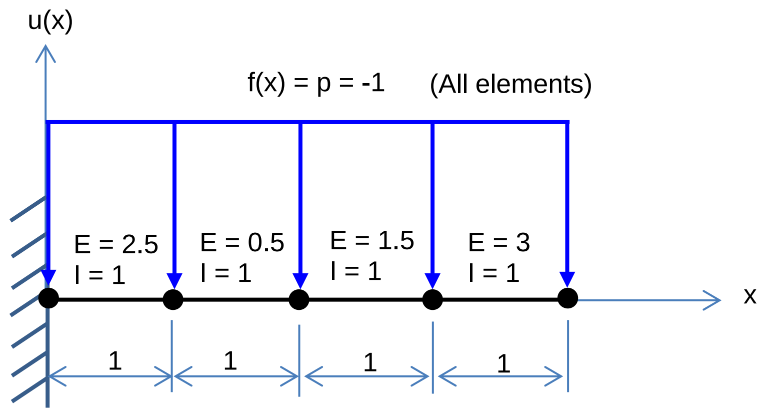
Nastran Dat file
http:www.3dwhiffletree.com/FEM/download/Chapter09-3.DAT

EXAMPLE 4 (VARIABLE E)

We change E values this time. Everything else is same as Example 1. Therefore,

$$\begin{cases} E_1 = 2.5 \\ E_2 = 0.5 \\ E_3 = 1.5 \\ E_4 = 3 \\ I_1 = I_2 = I_3 = I_4 = 1 \\ p_1 = p_2 = p_3 = p_4 = -1 \\ h_1 = h_2 = h_3 = h_4 = 1 \qquad \text{(Therefore, L = 4)} \end{cases}$$

The figure below summarizes the problem.



FEM Solution

Enter E values for each element.

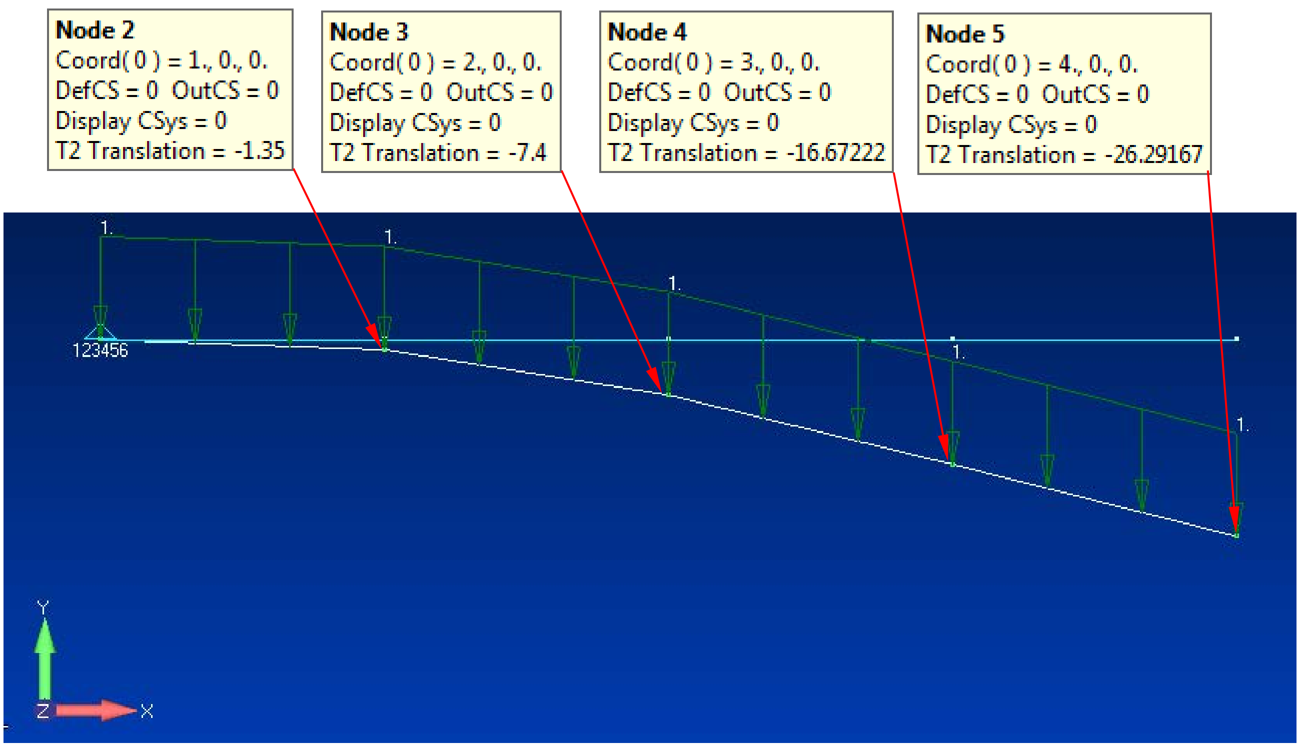
Element	1	2	3	4
E	2.5	0.5	1.5	3
I	1	1	1	1
h	1	1	1	1
p	-1	-1	-1	-1

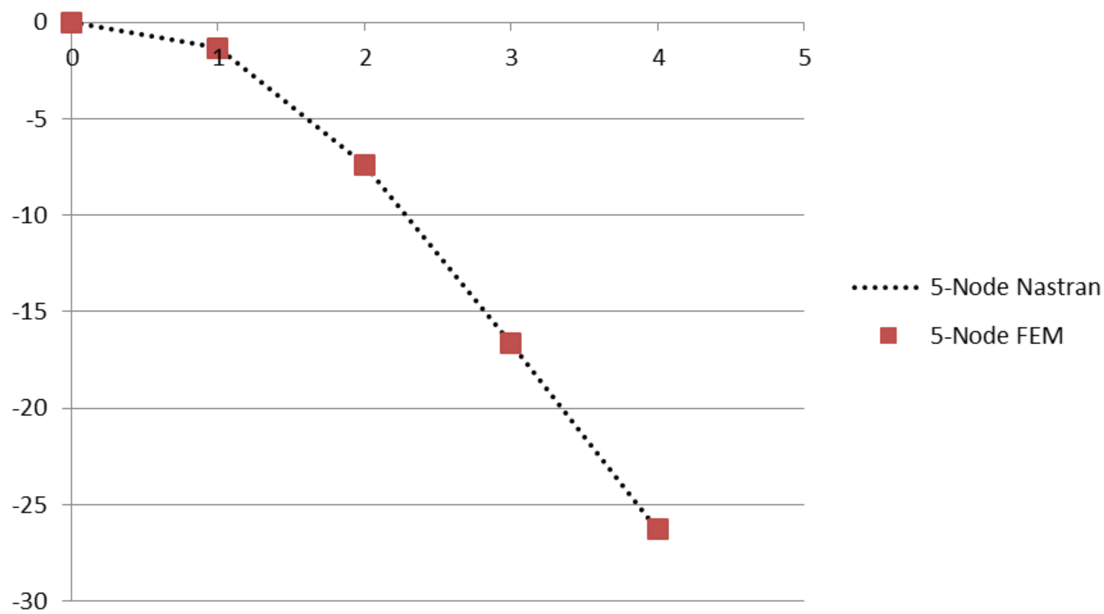
The Excel spreadsheet will automatically build a new stiffness matrix and give you this result.

$$d = \begin{array}{ll} \left\{ \begin{array}{r} -1.35 \\ -2.467 \\ -7.4 \\ -8.8 \\ -16.67 \\ -9.578 \\ -26.29 \\ -9.633 \end{array} \right\} & \begin{array}{l} \leftarrow d_2 \\ \\ \leftarrow d_3 \\ \\ \leftarrow d_4 \\ \\ \leftarrow d_5 \\ \\ \end{array} \end{array}$$

<u>Nastran Solution</u>

Since we cannot find an exact solution for such problem, again I want to compare our FEM results with Nastran solution. Below shows the results.





You can download both Excel file and Nastran files from the following links.

Excel File:
http:www.3dwhiffletree.com/FEM/download/Chapter09-4.xlsx

FEMAP neutral file (if you have FEMAP)
http:www.3dwhiffletree.com/FEM/download/Chapter09-4.NEU

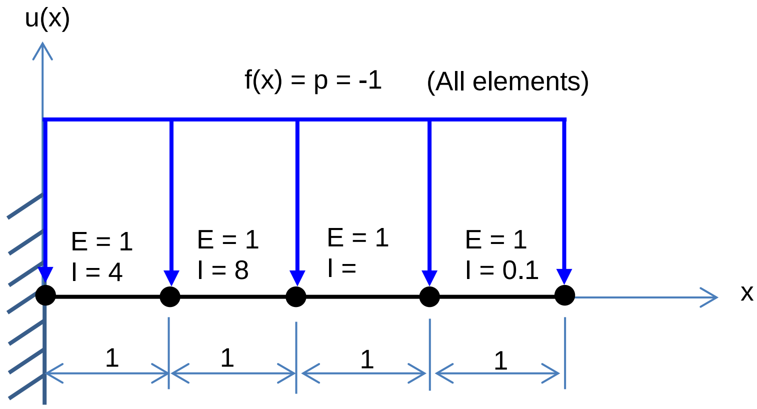
Nastran Dat file
http:www.3dwhiffletree.com/FEM/download/Chapter09-4.DAT

EXAMPLE 5 (VARIABLE I)

We change I vlaues this time. Everything else is same as Example 1. Therefore,

$$\begin{cases} E_1 = E_2 = E_3 = E_4 = 1 \\ I_1 = 4 \\ I_2 = 8 \\ I_3 = 0.25 \\ I_4 = 0.1 \\ p_1 = p_2 = p_3 = p_4 = -1 \\ h_1 = h_2 = h_3 = h_4 = 1 \qquad \text{(Therefore, L = 4)} \end{cases}$$

The figure below summarizes the problem.



FEM Solution

Enter I values for each element.

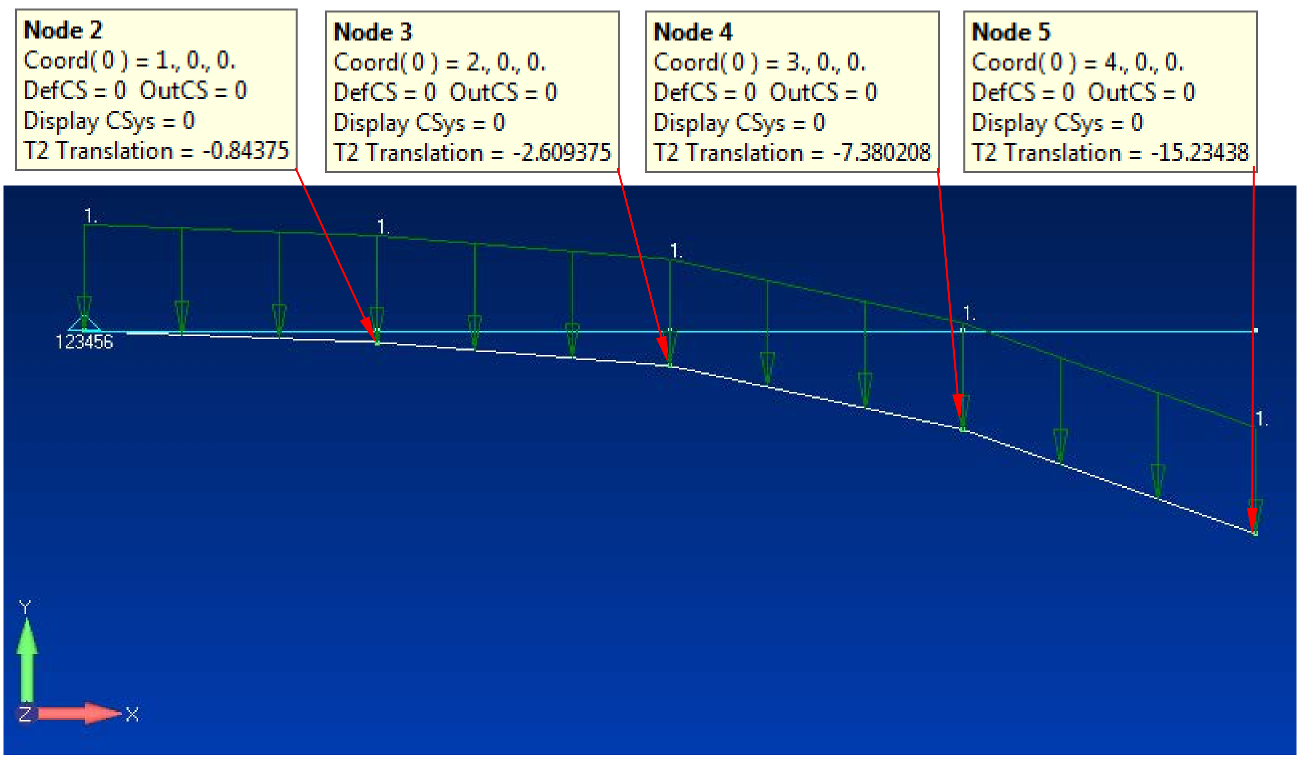
Element	1	2	3	4
E	1	1	1	1
I	4	8	0.25	0.1
h	1	1	1	1
p	-1	-1	-1	-1

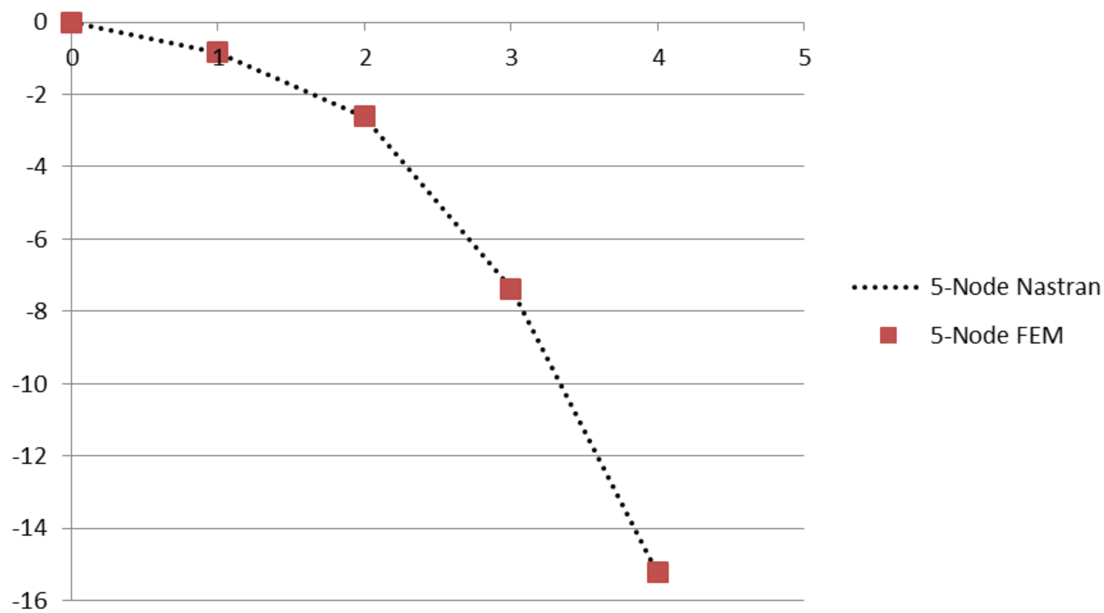
The Excel spreadsheet will automatically build a new stiffness matrix and give you this result.

$$d=\begin{array}{cl} \left\{\begin{array}{r} -0.844 \\ -1.542 \\ -2.609 \\ -1.938 \\ -7.38 \\ -6.604 \\ -15.23 \\ -8.271 \end{array}\right\} & \begin{array}{l} \leftarrow d_2 \\ \\ \leftarrow d_3 \\ \\ \leftarrow d_4 \\ \\ \leftarrow d_5 \\ \\ \end{array} \end{array}$$

<u>Nastran Solution</u>

Since we cannot find an exact solution for such problem, again I want to compare our FEM results with Nastran solution. Below shows the results.





You can download both Excel file and Nastran files from the following links.

Excel File:
http:www.3dwhiffletree.com/FEM/download/Chapter09-5.xlsx

FEMAP neutral file (if you have FEMAP)
http:www.3dwhiffletree.com/FEM/download/Chapter09-5.NEU

Nastran Dat file
http:www.3dwhiffletree.com/FEM/download/Chapter09-5.DAT

EXAMPLE 6 (VARIABLE ALL)

This time we use all variables together that were used in Example 2 through 6.

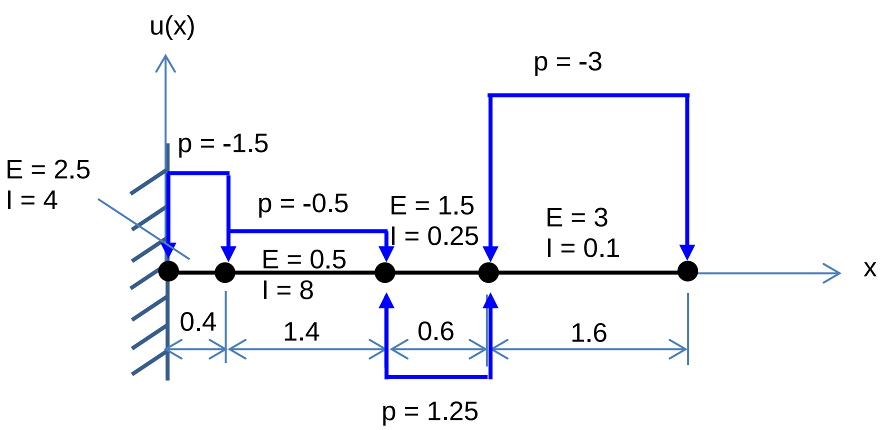
$$\begin{cases} E_1 = 2.5 \\ E_2 = 0.5 \\ E_3 = 1.5 \\ E_4 = 3 \end{cases}$$

$$\begin{cases} I_1 = 4 \\ I_2 = 8 \\ I_3 = 0.25 \\ I_4 = 0.1 \end{cases}$$

$$\begin{cases} p_1 = -1.5 \\ p_2 = -0.5 \\ p_3 = 1.25 \\ p_4 = -3 \end{cases}$$

$$\begin{cases} h_1 = 0.4 \\ h_2 = 1.4 \\ h_3 = 0.6 \\ h_4 = 1.6 \end{cases} \qquad \text{(Therefore, L = 4)}$$

The figure below summarizes the problem.



FEM Solution

Enter E, I, h, and p values for each element.

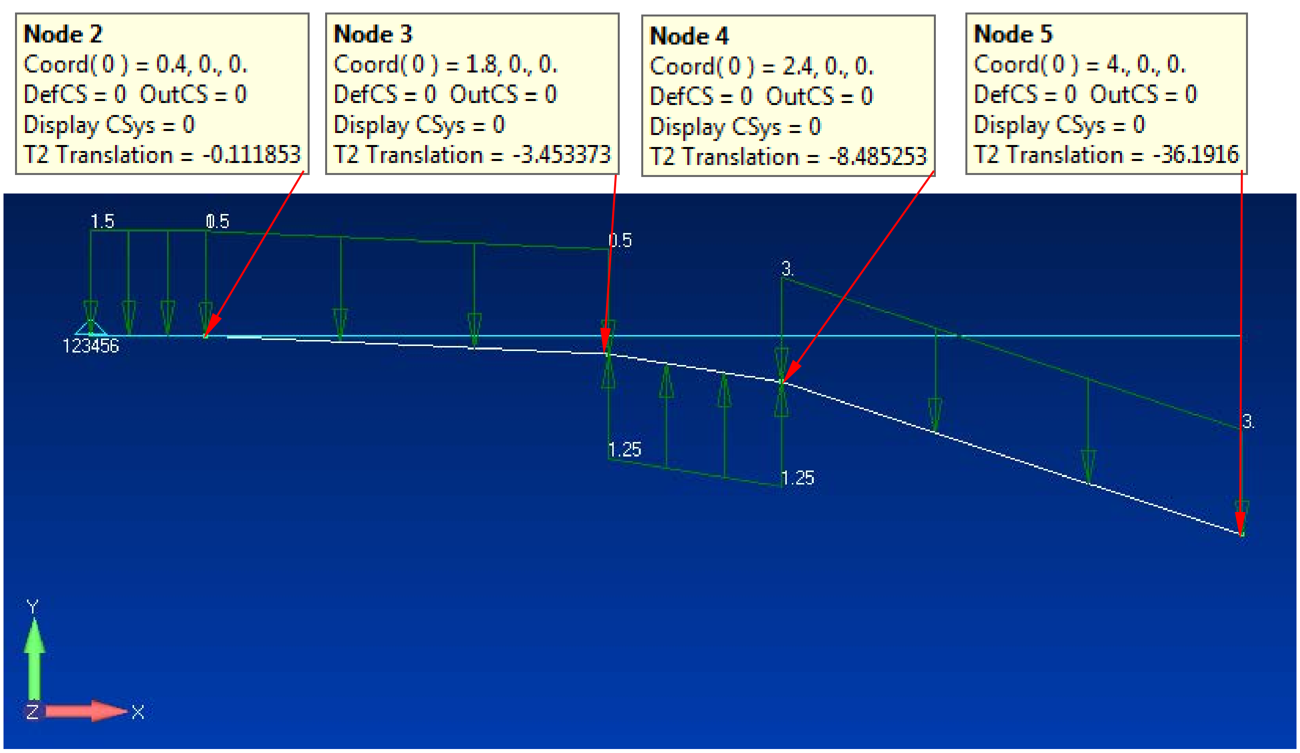
Element	1	2	3	4
E	2.5	0.5	1.5	3
I	4	8	0.25	0.1
h	0.4	1.4	0.6	1.6
p	-1.5	-0.5	1.25	-3

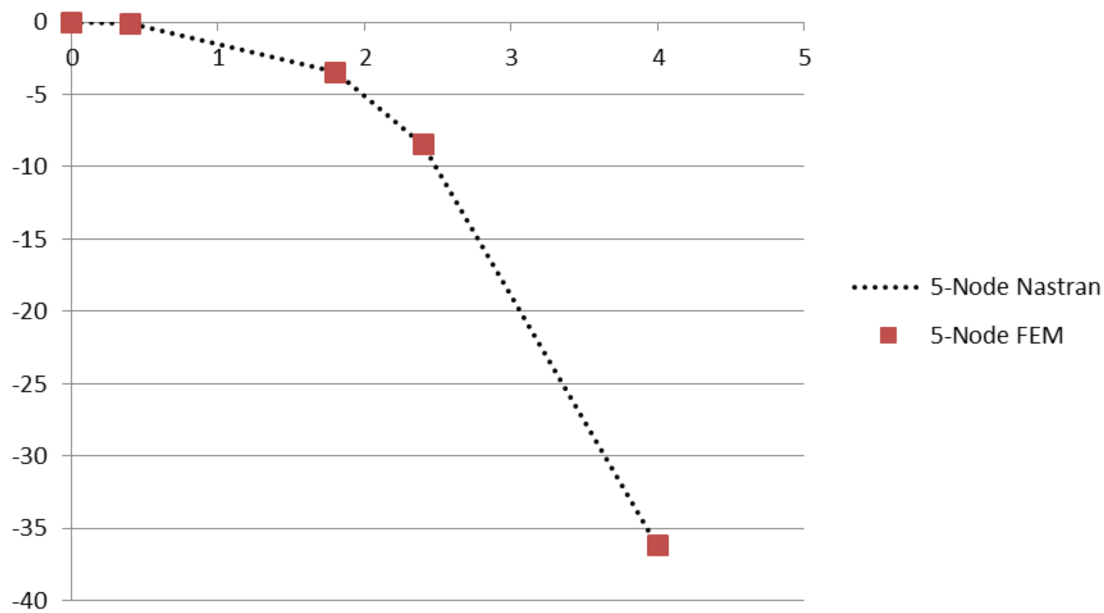
The Excel spreadsheet will automatically build a new stiffness matrix and give you this result.

$$d=\begin{Bmatrix} -0.112 \\ -0.546 \\ -3.453 \\ -3.868 \\ -8.485 \\ -12.2 \\ -36.19 \\ -19.02 \end{Bmatrix}\begin{array}{l} \leftarrow d_2 \\ \\ \leftarrow d_3 \\ \\ \leftarrow d_4 \\ \\ \leftarrow d_5 \\ \\ \end{array}$$

Nastran Solution

Since we cannot find an exact solution for such problem, again I want to compare our FEM results with Nastran solution. Below shows the results.





You can download both Excel file and Nastran files from the following links.

Excel File:
http:www.3dwhiffletree.com/FEM/download/Chapter09-6.xlsx

FEMAP neutral file (if you have FEMAP)
http:www.3dwhiffletree.com/FEM/download/Chapter09-6.NEU

Nastran Dat file
http:www.3dwhiffletree.com/FEM/download/Chapter09-6.DAT

MORE GENERAL FORMULA WITH F AND M IN ELEMENTS

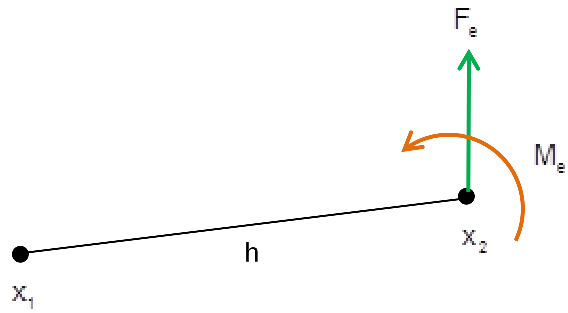
Now, consider the case if we have force (F) and Moment (M) in elements. From the previous pages, the global force vector was,

$$\bar{f} = \int_0^L f(x)w(x)dx + Fw(L) + M\frac{d}{dx}w(L) \qquad \text{per Eq.09-2}$$

And also we did

$$\bar{f} = \int_0^L f(x)N_B(x)dx + FN_B(L) + M\frac{d}{dx}N_B(x)\Big|_{x=L}$$

The term $N_B(L)$ and $\frac{d}{dx}N_B(x)\Big|_{x=L}$ are both 1 (because of boundary conditions and property of the shape functions). This is true for each element if there are F_e and M_e at the 2nd node.



Therefore, we simply just need to add F_e and M_e into the force vector.

Therefore, the matrix form of $\{f_a^e\} = [k_{ab}]\{d_b\}$ becomes,

$$\begin{Bmatrix} \frac{ph}{2} \\ \frac{ph^2}{12} \\ \frac{ph}{2} \\ -\frac{ph^2}{12} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ F_e \\ M_e \end{Bmatrix} = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix}\begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \end{Bmatrix} \qquad \text{..Eq.09-11}$$

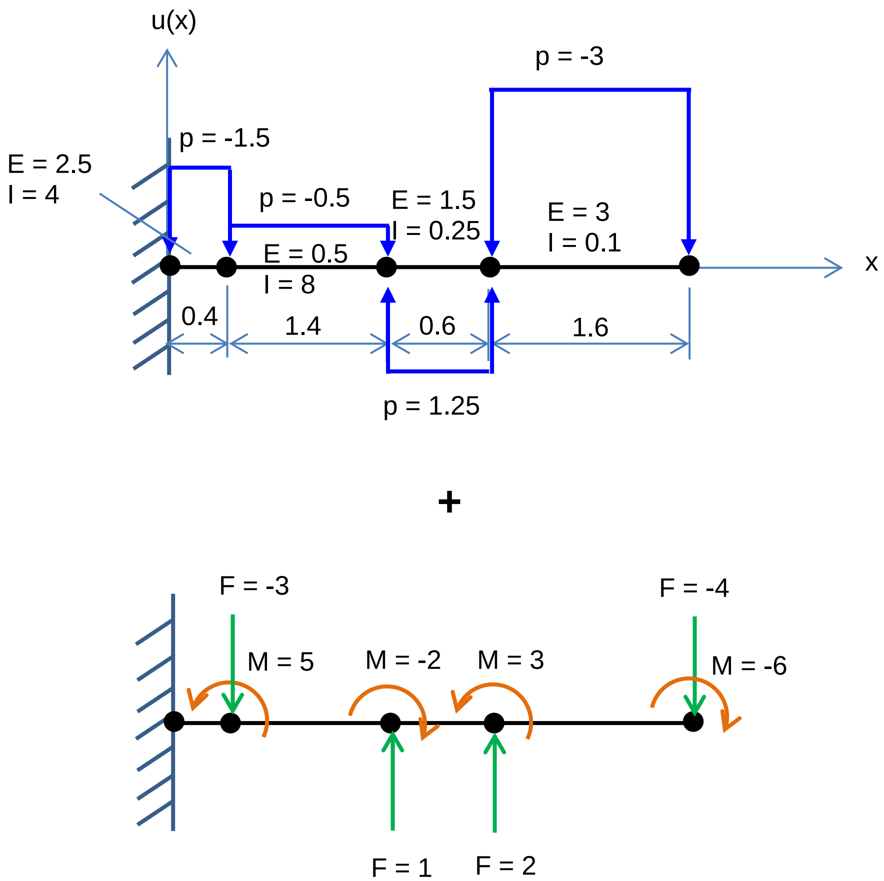
EXAMPLE 7 (VARIABLE ALL WITH FORCES AND MOMENTS)

In this example, we use the same values as Example 6 plus additional forces and moments as follows.

$$\begin{cases} F_1 = -3 \\ F_2 = 1 \\ F_3 = 2 \\ F_4 = -4 \end{cases}$$

$$\begin{cases} M_1 = 5 \\ M_2 = -2 \\ M_3 = 3 \\ M_4 = -6 \end{cases}$$

The figure below summarizes the problem.



FEM Solution

Create 2 more rows to the table so that you can enter F and M values for the each element.

Element	1	2	3	4
E	2.5	0.5	1.5	3
I	4	8	0.25	0.1
h	0.4	1.4	0.6	1.6
p	-1.5	-0.5	1.25	-3
F	-3	1	2	-4
M	5	-2	3	-6

Modify element force vectors to incorporate additional F and M.

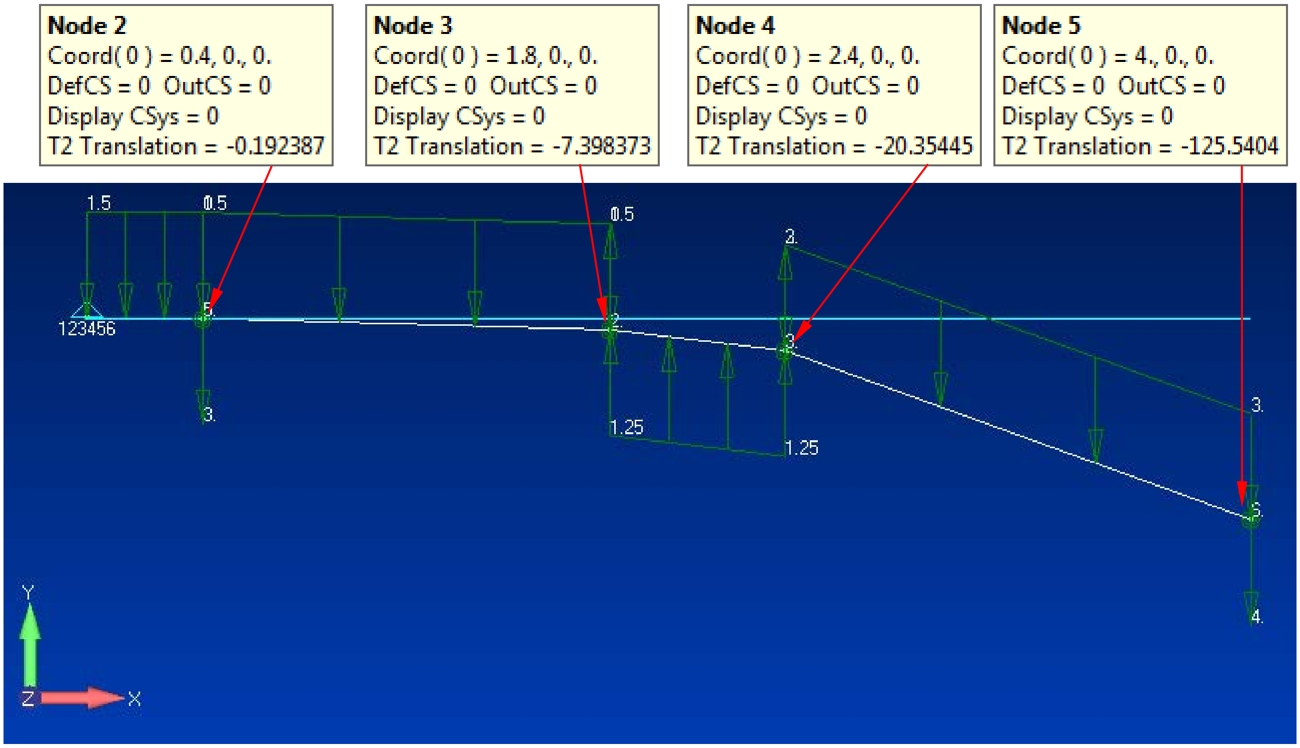
	A	B	C
45			
46		-0.3	
47	F1=	-0.02	
48		=_p1*_h1/2+_F1	
49		5.02	
50			
51			
52		-0.35	
53	F2=	-0.08167	
54		0.65	
55		-1.91833	
56			
57			
58		0.375	
59	F3=	0.0375	
60		2.375	
61		2.9625	
62			
63			
64		-2.4	
65	F4=	-0.64	
66		-6.4	
67		-5.36	

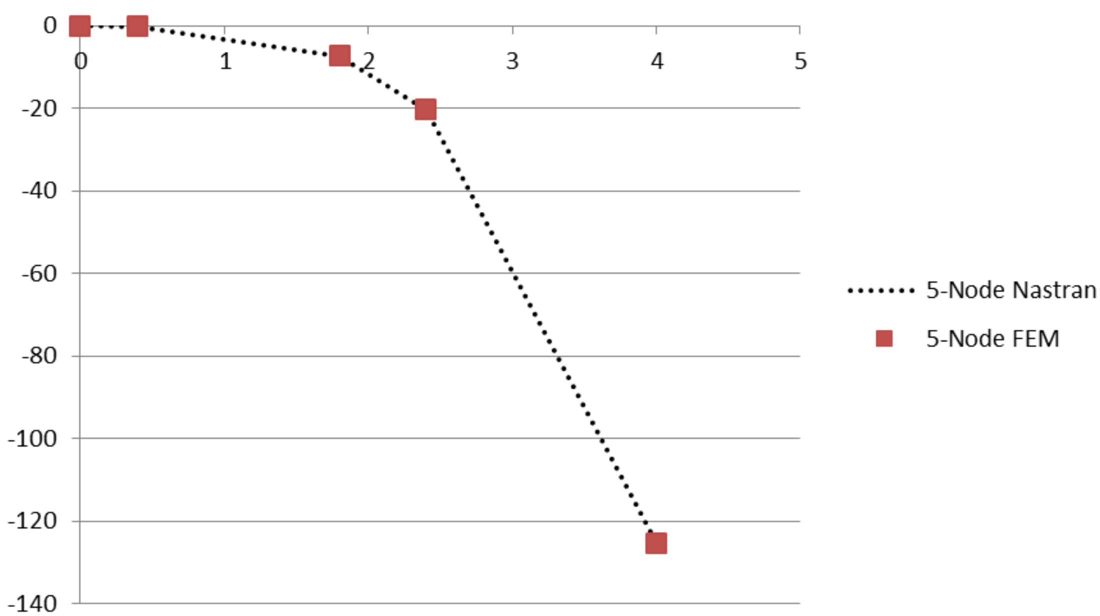
Rest of formulas will remain the same and the Excel spreadsheet gives you the result.

$$d=\begin{Bmatrix} -0.192 \leftarrow d_2 \\ -0.938 \\ -7.398 \leftarrow d_3 \\ -8.915 \\ -20.35 \leftarrow d_4 \\ -33.24 \\ -125.5 \leftarrow d_5 \\ -89.14 \end{Bmatrix}$$

Nastran Solution

Since we cannot find an exact solution for such problem, again I want to compare our FEM results with Nastran solution. Below shows the results.





You can download both Excel file and Nastran files from the following links.

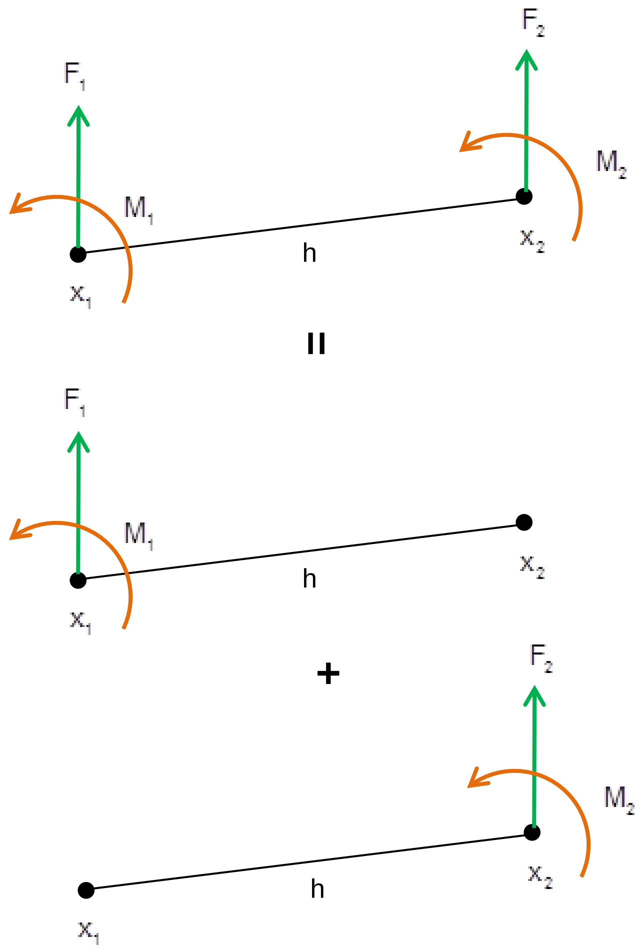
Excel File:
http:www.3dwhiffletree.com/FEM/download/Chapter09-7.xlsx

FEMAP neutral file (if you have FEMAP)
http:www.3dwhiffletree.com/FEM/download/Chapter09-7.NEU

Nastran Dat file
http:www.3dwhiffletree.com/FEM/download/Chapter09-7.DAT

FORCE AND MOMENT AT BOTH ENDS

If there are forces and moments at both ends of element, what will we get? The answer is a simple summation.



The previous example was intentionally done with force and moment at just one end to avoid confusion which node forces and moments belong to.

Therefore, we simply need to add F_1 and M_1 in the first 2 rows of force vectors (F_2 and M_2 in the last 2 rows which is same as Example 7).

$$\begin{Bmatrix} \dfrac{ph}{2} \\ \dfrac{ph^2}{12} \\ \dfrac{ph}{2} \\ -\dfrac{ph^2}{12} \end{Bmatrix} + \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \begin{Bmatrix} d_1 \\ \theta_1 \\ d_2 \\ \theta_2 \end{Bmatrix} \quad \text{..Eq.09-12}$$