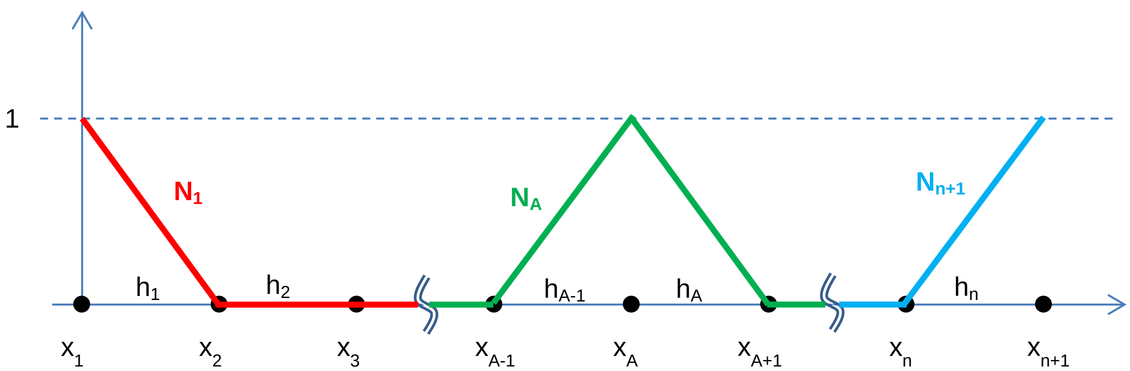
CHAPTER 08: PIECE-WISE LINEAR FINITE ELEMENT SPACE

GLOBAL COORDINATE SYSTEM

Let's review piece-wise linear shape function. A sketch of the shape function is shown below.



The shape function is defined as follows:

$$A = 1 \qquad N_A(x) = \begin{cases} \dfrac{x_2 - x}{h_1} & \text{for } x_1 \le x \le x_2 \\ 0 & \text{elsewhere} \end{cases}$$

$$2 \le A \le n \qquad N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{h_{A-1}} & \text{for } x_{A-1} \le x \le x_A \\ \dfrac{x_{A+1} - x}{h_A} & \text{for } x_A \le x \le x_{A+1} \\ 0 & \text{elsewhere} \end{cases}$$

$$A = n+1 \qquad N_A(x) = \begin{cases} \dfrac{x - x_n}{h_n} & \text{for } x_n \le x \le x_{n+1} \\ 0 & \text{elsewhere} \end{cases}$$

You may have recognized that the above equation for $N_A(x)$ only shows the slope. (The intercept at x_1 is omitted.) In fact, it is ok because later we are only focusing on the stiffness matrix which is calculated from the derivative of the slope of the shape function (and thus, any constants will drop form the equation).

Here, "h" is a mesh size. That is, in general,

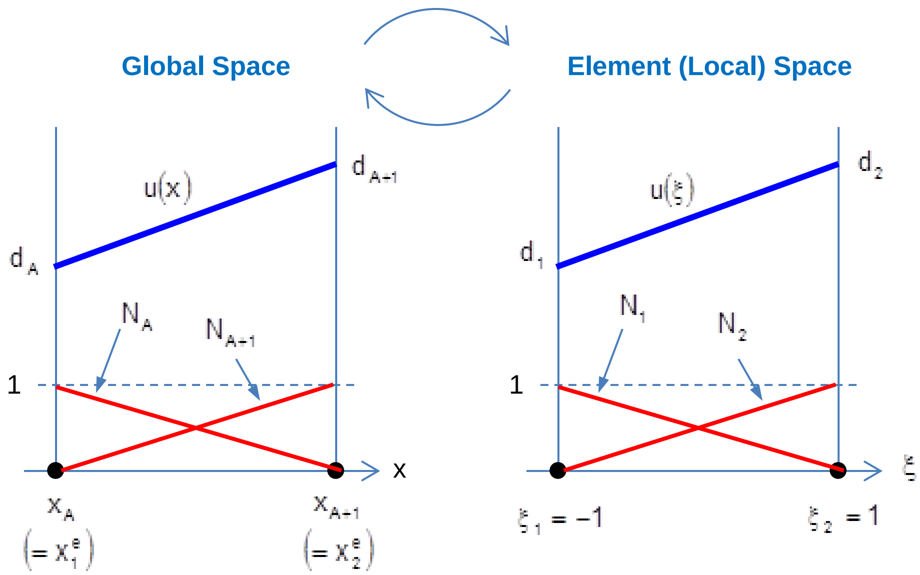
$$h_A = x_{A+1} - x_A \qquad \text{.......Eq.08-1}$$

The subdomain on h_A is called **finite element space**, or simply **element**.

ELEMENT POINT OF VIEW

So far we have been talking everything in global space (or global coordinate system). Now let's introduce **Element Space** (or element coordinate system). Element space is also called **local coordinate system**.

Each element domain is defined using ξ_1 and ξ_2, and have value -1 and 1, respectively. The following shows comparison of Global and Element spaces.



	Global Space	Element (Local) Space
Domain:	$[x_A, x_{A+1}]$	$[\xi_1, \xi_2] = [-1, 1]$
Nodes:	$\{x_A, x_{A+1}\}$	$\{\xi_1, \xi_2\}$
Degree of Freedom:	$\{d_A, d_{A+1}\}$	$\{d_1, d_2\}$
Shape Functions:	$\{N_A, N_{A+1}\}$	$\{N_1, N_2\}$
Interpolate Function:	$u(x) = N_A(x)d_A + N_{A+1}(x)d_{A+1}$	$u(\xi) = N_1(\xi)d_1 + N_2(\xi)d_2$

ξ is a Latin character for x.

The element local domain is always from **-1** to **+1**.
This makes calculation much easier.

Conversion between global and element space is called **mapping**.

The interpolate function is what we called approximate solution.

SHAPE FUNCTION IN ELEMENT SPACE

The shape function in terms of local coordinate system (in terms of ξ) can be shown as follows:

$$N_1(\xi) = -\frac{1}{2}\xi + \frac{1}{2}$$

$$N_2(\xi) = \frac{1}{2}\xi + \frac{1}{2}$$

Or, the above two equations can be expressed in one equation as follows:

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \qquad a = 1, 2 \quad \text{.........Eq.08-2}$$

$\xi_1 = -1$ and $\xi_2 = 1$

LOCAL AND GLOBAL LOCATION

The local location in terms of global variable (that is, $\xi(x)$) can be expressed, using a constants c_1 and c_2, as,

$$\xi(x) = c_1 + c_2 \cdot x$$

The constant c_1 and c_2 can be determined by

$$\begin{cases} \xi(x_A) = c_1 + c_2 \cdot x_A = -1 & \text{.........Eq.08-3} \\ \xi(x_{A+1}) = c_1 + c_2 \cdot x_{A+1} = 1 & \text{.........Eq.08-4} \end{cases}$$

Solving for c_1 from Eq.08-3 and inserting it into Eq.08-4, we get

$$(-1 - c_2 \cdot x_A) + c_2 \cdot x_{A+1} = 1$$

$$\Rightarrow c_2 \cdot (x_{A+1} - x_A) = 2$$

$$\Rightarrow c_2 = \frac{2}{x_{A+1} - x_A}$$

Therefore, inserting this c_2 into Eq.08-3,

$$c_1 + \frac{2}{x_{A+1} - x_A} \cdot x_A = -1$$

$$\Rightarrow c_1 = -\frac{2}{x_{A+1} - x_A} \cdot x_A - 1 = -\frac{2x_A}{x_{A+1} - x_A} - \frac{x_{A+1} - x_A}{x_{A+1} - x_A} = -\left(\frac{2x_A}{x_{A+1} - x_A} + \frac{x_{A+1} - x_A}{x_{A+1} - x_A}\right) = -\frac{x_A + x_{A+1}}{x_{A+1} - x_A}$$

Therefore, inserting c_1 and c_2,

$$\xi(x) = -\frac{x_A + x_{A+1}}{x_{A+1} - x_A} + \frac{2}{x_{A+1} - x_A} \cdot x$$

$$\Rightarrow \xi(x) = \frac{2x - x_A - x_{A+1}}{x_{A+1} - x_A}$$

Here, since $h_A = x_{A+1} - x_A$ (Eq.08-1), we get

$$\xi(x) = \frac{2x - x_A - x_{A+1}}{h_A} \quad \text{......Eq.08-5}$$

If we solving Eq.08-5 for x, we can obtain global location in terms of local variable, that is, $x(\xi)$.

$$\Rightarrow h_A \xi = 2x - x_A - x_{A+1}$$

Therefore,

$$x(\xi) = \frac{h_A \xi + x_A + x_{A+1}}{2} \quad \text{......Eq.08-6}$$

In local coordinate system, $h_A = x_{A+1} - x_A$, the Eq.08-6 can be written as

$$x(\xi) = \frac{(x_{A+1} - x_A)\xi + x_A + x_{A+1}}{2}$$

Further, using $x_A = x_1^e$, and $x_{A+1} = x_2^e$, we can express it as

$$x(\xi) = x^e(\xi) = \frac{(x_2^e - x_1^e)\xi + x_1^e + x_2^e}{2} = \frac{1}{2}(1-\xi)x_1^e + \frac{1}{2}(1+\xi)x_2^e$$

Remember, from Eq.08-2, we have

$$N_a(\xi) = \frac{1}{2}(1 + \xi_a \xi) \qquad a = 1, 2$$

Therefore, the above $x^e(\xi)$ can be shown as

$$x^e(\xi) = \underbrace{\frac{1}{2}(1-\xi)}_{N_1(\xi)} x_1^e + \underbrace{\frac{1}{2}(1+\xi)}_{N_2(\xi)} x_2^e = N_1 x_1^e + N_2 x_2^e$$

Finally, in a simple form, the above equation can be written as

$$x^e(\xi) = \sum_{a=1}^{2} N_a(\xi) x_a^e \quad \text{......Eq.08-7}$$

OTHER IMPORTANT EQUATIONS FOR FUTURE REFERENCES

Let's find some other important relations for future purpose.

Taking the derivative of Eq.08-2 with respect to ξ,

$$\frac{d}{d\xi}N_a = \frac{d}{d\xi}\left[\frac{1}{2}(1+\xi_a\xi)\right] = \frac{\xi_a}{2} = \frac{(-1)^a}{2}$$

Thus,

$$\frac{d}{d\xi}N_a = \frac{(-1)^a}{2} \qquad \text{Eq.08-8}$$

Taking the derivative of Eq.08-6 with respect to ξ,

$$\frac{d}{d\xi}x^e = \frac{d}{d\xi}\left(\frac{h_A\xi + x_A + x_{A+1}}{2}\right) = \frac{d}{d\xi}\left(\frac{h^e\xi + x_1^e + x_2^e}{2}\right) = \frac{h^e}{2}$$

Thus,

$$\frac{d}{d\xi}x^e = \frac{h^e}{2} \qquad \text{Eq.08-9}$$

Taking the derivative of Eq.08-5 with respect to x,

$$\frac{d}{dx}\xi = \frac{d}{dx}\left(\frac{2x - x_A - x_{A+1}}{h_A}\right)$$

In terms of element space ξ^e,

$$\frac{d}{dx}\xi^e = \frac{d}{dx}\left(\frac{2x - x_1^e - x_2^e}{h^e}\right) = \frac{2}{h^e}$$

Thus,

$$\frac{d}{dx}\xi^e = \frac{2}{h^e} \qquad \text{Eq.08-10}$$

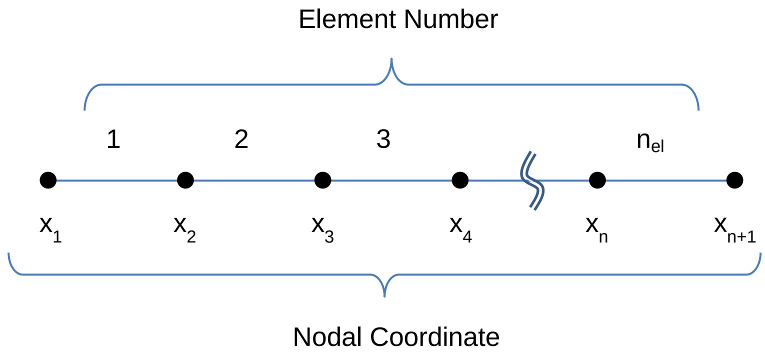
From Eq.08-9 and Eq.08-10, we also can say,

$$\frac{d}{dx}\xi^e = \left(\frac{d}{d\xi}x^e\right)^{-1} \qquad \text{Eq.08-11}$$

$$\frac{d}{dx}\xi^e = \frac{2}{h^e} = \left(\frac{h^e}{2}\right)^{-1} = \left(\frac{d}{d\xi}x^e\right)^{-1}$$

ASSEMBLING GLOBAL MATRIX FORMS

Let's review a global matrix form for the following general nodes and elements.



The force vectors and stiffness matrix in a global matrix form (Eq.05-10) has the following sizes.

$$\underbrace{\{F_A\}}_{n \times 1} = \underbrace{[K_{AB}]}_{n \times n}\{d_B\}$$

The global integral can be shown as a sum of local integral over the element domain. Therefore, stiffness matrix and force vectors can be shown as,

$$[K_{AB}] = \sum_{e=1}^{n_{el}} [k_{ab}^e]$$

$$\{F_A\} = \sum_{e=1}^{n_{el}} \{f_a^e\}$$

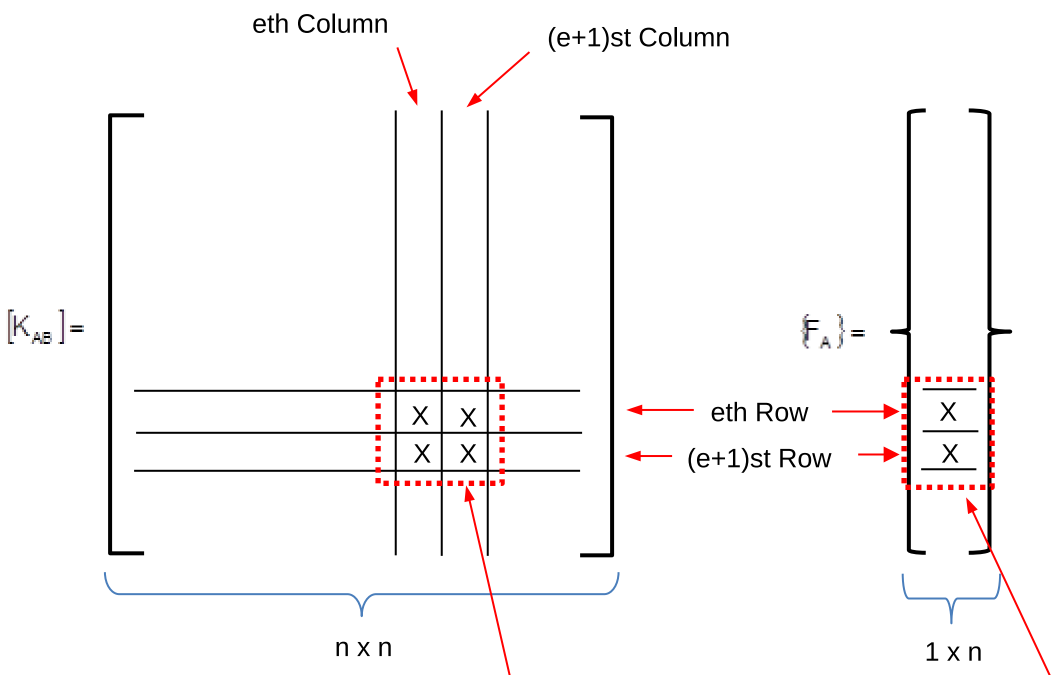
Size of local matrix and vector
$[k^e]$ = 2 x 2
$\{f^e\}$ = 2 x 1

,where $$[k_{ab}^e] = \int_{\Omega^e} \frac{d}{dx}N_a \frac{d}{dx}N_b dx$$ Eq.08-12

$$\{f_a^e\} = \int_{\Omega^e} N_a f dx + \begin{cases} \delta_{a1} h & \text{for } e = 1 \\ 0 & \text{for } e = 2, 3, \ldots, n_{el} - 1 \\ -\delta_{a2} g & \text{for } e = n_{el} \end{cases}$$

$\Omega^e = [x_1^e, x_2^e]$ (**The domain of the element**)

The following figure depicts how in general the local components are assembled into global matrix.



$$[k^e_{ab}] = \begin{bmatrix} k^e_{11} & k^e_{12} \\ k^e_{21} & k^e_{22} \end{bmatrix} \qquad \{f^e_a\} = \begin{Bmatrix} f^e_1 \\ f^e_2 \end{Bmatrix}$$

CHANGE INTEGRAL FROM GLOBAL TO ELEMENTAL COORDINATE SYSTEM

Now, let's review two general mathematical formulas.

First, the **Change of Variable Formula**, which we want to convert x (global coordinate) in terms of ξ (local coordinate), is given in the following formula:

$$\int_{x_1}^{x_2} f(x)dx = \int_{\xi_1}^{\xi_2} f(x(\xi))\frac{d}{d\xi}x(\xi)d\xi$$

> By the way, in Change of Variable Formula, the term $\frac{d}{d\xi}x(\xi)$ is known as **Jacobian Determinant**.
>
> We'll talk about this later.

Second, the **Chain Rule** is given in the following formula:

$$\frac{\partial}{\partial\xi}f(x(\xi)) = \frac{\partial}{\partial x}f(x(\xi))\frac{\partial}{\partial\xi}x(\xi)$$

By knowing the above two formulas, now let's convert the stiffness matrix from global coordinate system to a local, element coordinate system.

We start from Eq.08-12 as

$$\left[k_{ab}^e\right] = \int_{\Omega^e} \frac{d}{dx}N_a \frac{d}{dx}N_b dx \qquad \text{(Eq.08-12)}$$

By using a change of variable formula, it becomes

$$\Rightarrow \left[k_{ab}^e\right] = \int_{-1}^{+1} \frac{d}{dx}N_a(x(\xi))\frac{d}{dx}N_b(x(\xi))\frac{d}{d\xi}x(\xi)d\xi$$

Using change of variable formula,

$$\int_{\Omega^e}(\quad)dx \Rightarrow \int_{-1}^{+1}(\quad)\frac{d}{d\xi}x(\xi)d\xi$$

Then, by using chain rule,

$$\Rightarrow \left[k_{ab}^e\right] = \int_{-1}^{+1} \left[\frac{\frac{d}{d\xi}N_a(\xi)}{\frac{d}{d\xi}x(\xi)}\right]\left[\frac{\frac{d}{dx}N_b(\xi)}{\frac{d}{d\xi}x(\xi)}\right]\frac{d}{d\xi}x(\xi)d\xi$$

$$= \int_{-1}^{+1} \frac{d}{d\xi}N_a(\xi)\left[\frac{d}{d\xi}x(\xi)\right]^{-1}\frac{d}{d\xi}N_b(\xi)\left[\frac{d}{d\xi}x(\xi)\right]^{-1}\frac{d}{d\xi}x(\xi)d\xi$$

cancelled out

$$= \int_{-1}^{+1} \frac{d}{d\xi}N_a(\xi)\frac{d}{d\xi}N_b(\xi)\left[\frac{d}{d\xi}x(\xi)\right]^{-1}d\xi$$

Using chain rule,

$$\frac{d}{d\xi}N(x(\xi)) = \frac{d}{dx}N(x(\xi))\frac{d}{d\xi}x(\xi)$$

Therefore,

$$\frac{d}{dx}N(x(\xi)) = \frac{d}{dx}N(\xi) = \frac{\frac{d}{d\xi}N(x(\xi))}{\frac{d}{d\xi}x(\xi)}$$

Now, recall the following equations from previous pages.

$$\frac{d}{d\xi}N_a = \frac{(-1)^a}{2} \qquad \text{(Eq.08-8)}$$

$$\frac{d}{d\xi}x^e = \frac{h^e}{2} \qquad \text{(Eq.08-9.)}$$

Substituting these equations, then, the stiffness matrix equation becomes,

$$\left[k_{ab}^e\right] = \int_{-1}^{+1} \underbrace{\frac{d}{d\xi}N_a(\xi)}_{\frac{(-1)^a}{2}}\underbrace{\frac{d}{d\xi}N_b(\xi)}_{\frac{(-1)^b}{2}}\underbrace{\left[\frac{d}{d\xi}x(\xi)\right]^{-1}}_{\left(\frac{h^e}{2}\right)^{-1}}d\xi = \int_{-1}^{+1}\frac{(-1)^a}{2}\cdot\frac{(-1)^b}{2}\cdot\frac{2}{h^e}d\xi = \int_{-1}^{+1}\frac{(-1)^{a+b}}{2h^e}d\xi = \frac{(-1)^{a+b}}{2h^e}\xi\Big|_{-1}^{+1} = \frac{(-1)^{a+b}}{h^e}$$

Therefore, each component of $\left[k_{ab}^e\right]$ is,

$$k_{11}^e = \frac{(-1)^{1+1}}{h^e} = \frac{1}{h^e}$$

$$k_{12}^e = \frac{(-1)^{1+2}}{h^e} = -\frac{1}{h^e}$$

$$k_{21}^e = \frac{(-1)^{2+1}}{h^e} = -\frac{1}{h^e}$$

$$k_{22}^e = \frac{(-1)^{2+2}}{h^e} = \frac{1}{h^e}$$

Therefore, we can express $\left[k_{ab}^e\right]$ in a very simple form as follows:

$$\left[k_{ab}^e\right] = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{.....Eq.08-13}$$

EXAMPLE

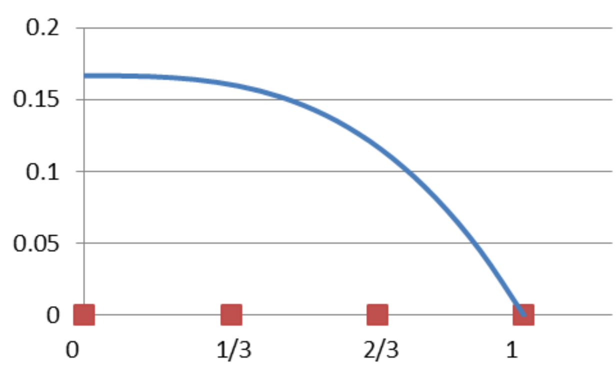
Let's do an example problem of using local stiffness matrix and assembling into a global stiffness matrix. We'll be using the same problem as we did in Chapter 07.

In Eq.07-1, we had the following problem,

$$\begin{cases} \dfrac{d^2}{dx^2}u + x = 0 \quad \text{on} \quad \Omega = [0,1] \\ u(1) = 0 \\ -\dfrac{d}{dx}u(0) = 0 \end{cases}$$

And, we approximated at the following points.

$$x = 0\,,\ \frac{1}{3}\,,\ \frac{2}{3}\,,\ 1$$



The global stiffness matrix was,

$$[K_{AB}] = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \qquad \text{(Eq.07-4)}$$

Now, let's look at the element space.

The number of element is,

$$n_{el} = 3$$

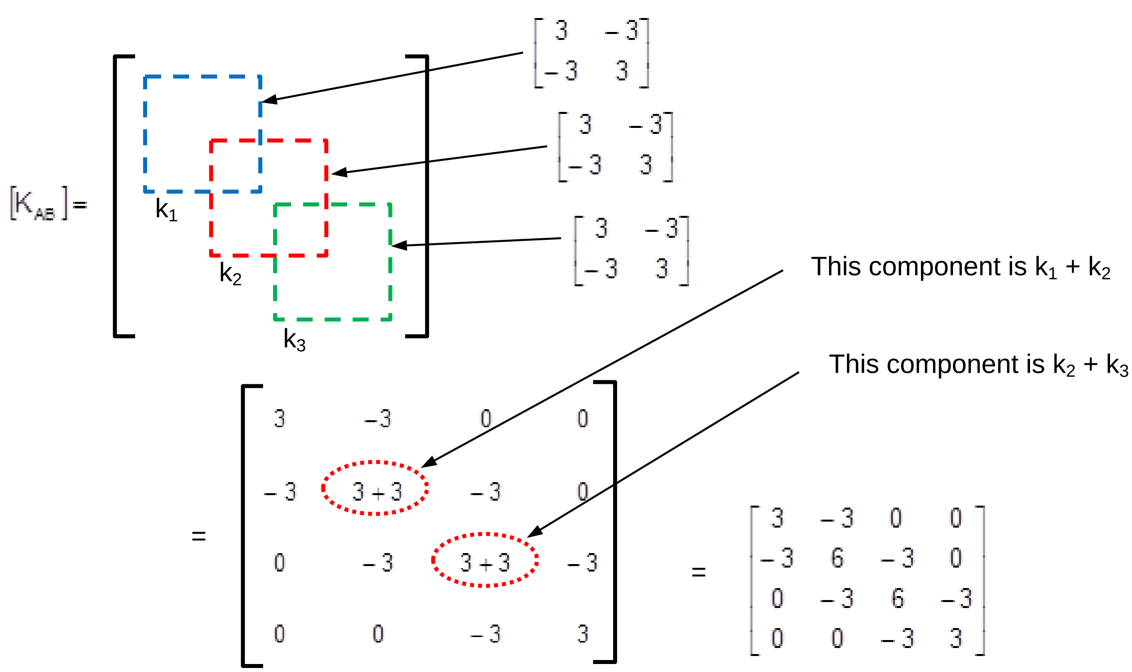
Mesh size of each elements is all equal to $\frac{1}{3}$, that is,

$$h^e = h_1 = h_2 = h_3 = \frac{1}{3}$$

Therefore, each local stiffness matrix is all equal to,

$$[k^e] = [k_1] = [k_2] = [k_3] = \frac{1}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{\left(\frac{1}{3}\right)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

The global stiffness matrix is an assembly of these local stiffness matrixes as the following figure.



Thus, we had the same global stiffness matrix as Eq.07-4; however, it was much easily obtained without doing any integral calculations this time.