CHAPTER 07: EXAMPLE PROBLEM (1-DIMENSIONAL, 3-DEGREE OF FREEDOM)

OBJECTIVE

Let's go over an example problem. This could be your first FEM problem solving it by hands. Object of this exercise is as follows:

- Solve FEM by hand
- Solve 1-dimensional, 3-degree of freedom PDE problem
- Use **Piece-wise Linear Shape Functions**
- Use global coordinate system only
- Solve **Matrix Form** of FEM using Excel spreadsheet and plot the results

PROBLEM

We'll be using the same differential equation introduced in chapter 5, that is,

Suppose we have the followings:

f: a given function in a domain $\Omega = [0,1]$ that is real numbers

g, h: constants

Then, suppose we want to solve the following differential equation for u, that is,

$$\begin{cases} \dfrac{d^2}{dx^2}u + f = 0 \ \text{ on } \ \Omega = [0,1] \\ u(1) = g \\ -\dfrac{d}{dx}u(0) = h \end{cases}$$

However, in this problem, let's assign f, g, and h as follows:

$$f = x$$

$$u(1) = g = 0$$

$$-\frac{d}{dx}u(0) = h = 0$$

Therefore, the actual problem is as follows:

$$\begin{cases} \dfrac{d^2}{dx^2}u + x = 0 \ \text{ on } \ \Omega = [0,1] \quad \text{.......Eq.07-1} \\ u(1) = 0 \\ -\dfrac{d}{dx}u(0) = 0 \end{cases}$$

This makes the differential equation much simple. Even though the problem is overly simplified compared to the real structural engineering problem, the basis for solving the FEM is essentially the same.

EXACT SOLUTION

This differential equation problem is simple enough so that we can luckily find an exact solution. So, let's find an exact solution first. The Eq.07-1 is,

$$\frac{d^2}{dx^2}u = -x$$

Integrate it once,

$$\Rightarrow \frac{d}{dx}u = -\frac{1}{2}x^2 + C_1 \quad \text{...Eq.07-2}$$

Applying the boundary condition $-\frac{d}{dx}u(0) = 0$, then,

$$\Rightarrow \cancelto{0}{\frac{d}{dx}u(1)} = -\frac{1}{2}(1)^2 + C_1 = 0$$

$$\Rightarrow C_1 = 0$$

Inserting C_1=0 into Eq.07-2 and integrate it once again,

$$\Rightarrow u = -\frac{1}{6}x^3 + \cancelto{0}{C_1} + C_2$$

Applying the boundary condition $u(1) = 0$, then,

$$\Rightarrow \cancelto{0}{u(1)} = -\frac{1}{6}(1)^3 + C_2 = 0$$

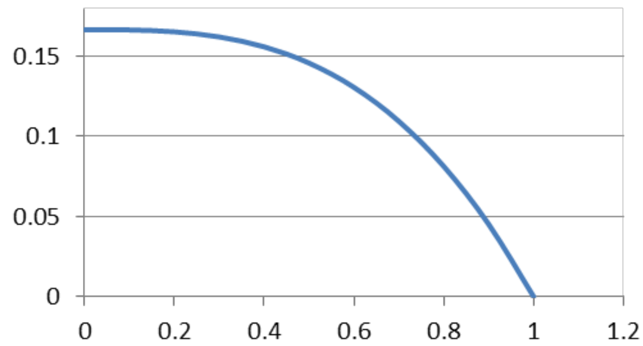
Therefore,

$$\Rightarrow C_2 = \frac{1}{6}$$

So, the exact solution is,

$$u(x) = -\frac{1}{6}x^3 + \frac{1}{6} \quad \text{...Eq.07-3}$$

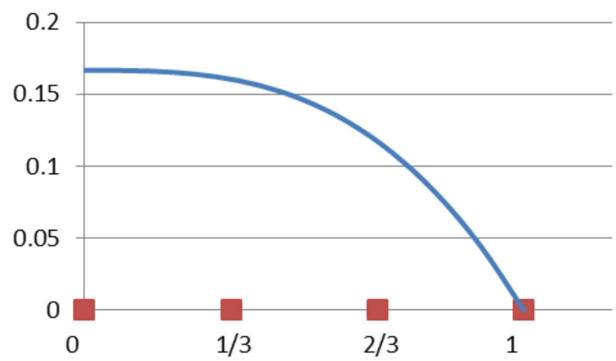
The curve of the exact solution looks like this:



PIECE-WISE LINEAR SHAPE FUNCTION

We want to approximate the solution at the following 4 locations (shown as red dot in the figure below).

$$x = 0\,,\ \frac{1}{3}\,,\ \frac{2}{3}\,,\ 1$$



Therefore, we have 4 nodes and 3 elements. The number of n (used in the equations derived in Chapter 5) is 3.

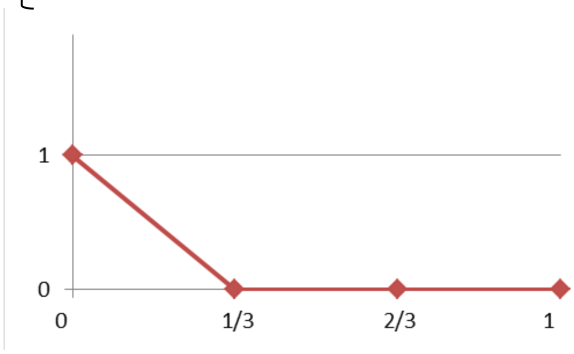
Node ID	x	A
1	0	1
2	1/3	2
3	2/3	3 = n
4	1	4= n+1

In this problem, we'll be using **Piece-wise Linear Shape Function**. The equation of the shape function and also derivative of the shape function are as follows:

$\underline{N_1}$

$$N_1(x) = \begin{cases} -3x + 1 & \text{for } 0 \le x \le \frac{1}{3} \\ 0 & \text{for } \frac{1}{3} \le x \le 1 \end{cases}$$

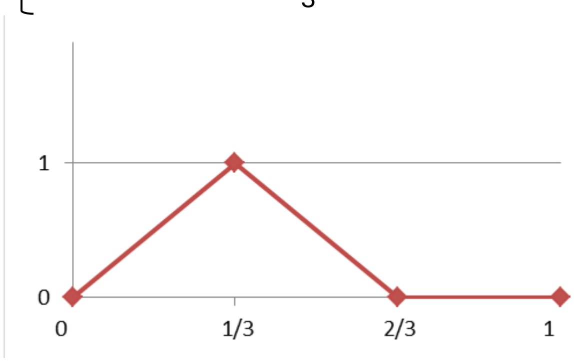
$$\frac{d}{dx}N_1(x) = \begin{cases} -3 & \text{for } 0 \le x \le \frac{1}{3} \\ 0 & \text{for } \frac{1}{3} \le x \le 1 \end{cases}$$



$\underline{N_2}$

$$N_2(x) = \begin{cases} 3x & \text{for } 0 \le x \le \frac{1}{3} \\ -3x + 2 & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$

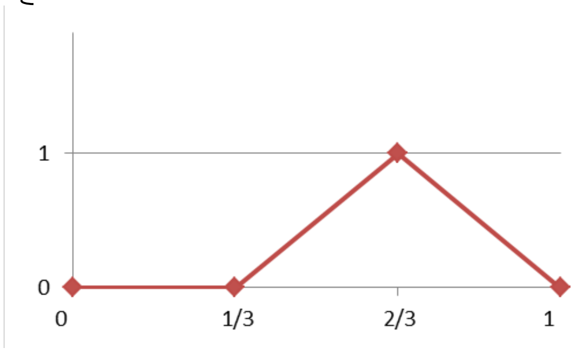
$$\frac{d}{dx}N_2(x) = \begin{cases} 3 & \text{for } 0 \le x \le \frac{1}{3} \\ -3 & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \text{for } \frac{1}{3} \le x \le 1 \end{cases}$$



$\underline{N_3}$

$$N_3(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{3} \\ 3x - 1 & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ -3x + 3 & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$

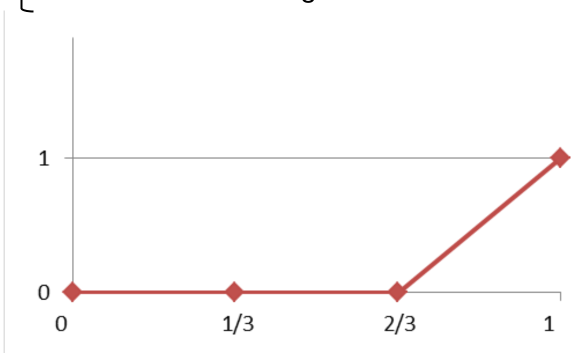
$$\frac{d}{dx} N_3(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{1}{3} \\ 3 & \text{for } \frac{1}{3} \le x \le \frac{2}{3} \\ -3 & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$



$\underline{N_4}$

$$N_4(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{2}{3} \\ 3x - 2 & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$

$$\frac{d}{dx} N_4(x) = \begin{cases} 0 & \text{for } 0 \le x \le \frac{2}{3} \\ 3 & \text{for } \frac{2}{3} \le x \le 1 \end{cases}$$



STIFFNESS MATRIX

Stiffness matrix for this PDE problem is given by Eq.05-8.

$$\tilde{K}_{AB} = [K_{AB}] = \int_0^1 \frac{d}{dx}N_A \frac{d}{dx}N_B dx$$

This stiffness matrix is for this given problem. Don't confuse it as if it is a general stiffness matrix.

In this problem,

A = 1, 2, 3, 4
B = 1, 2, 3, 4

This is 3-DOF problem, so you won't need to think of A=4 and B=4 (because the last displacement is given as boundary condition. However, let's assume for now that we have 4-DOF and we'll eliminate the known displace later.

Let's solve for each component of the stiffness matrix.

$$[K_{11}] = \int_0^1 \frac{d}{dx}N_1 \frac{d}{dx}N_1 dx = \int_0^{1/3}(-3)(-3)dx + \cancelto{0}{\int_{1/3}^{1}(0)(0)dx} = 9x\Big|_0^{1/3} = 9\left[\frac{1}{3} - 0\right] = 3$$

$$[K_{12}] = \int_0^1 \frac{d}{dx}N_1 \frac{d}{dx}N_2 dx = \int_0^{1/3}(-3)(3)dx + \cancelto{0}{\int_{1/3}^{2/3}(0)(-3)dx} + \cancelto{0}{\int_{2/3}^{1}(0)(0)dx} = -9x\Big|_0^{1/3} = (-9)\left[\frac{1}{3} - 0\right] = -3$$

$$[K_{13}] = \int_0^1 \frac{d}{dx}N_1 \frac{d}{dx}N_3 dx = \cancelto{0}{\int_0^{1/3}(-3)(0)dx} + \cancelto{0}{\int_{1/3}^{2/3}(0)(3)dx} + \cancelto{0}{\int_{2/3}^{1}(0)(-3)dx} = 0$$

$$[K_{14}] = \int_0^1 \frac{d}{dx}N_1 \frac{d}{dx}N_4 dx = \cancelto{0}{\int_0^{1/3}(-3)(0)dx} + \cancelto{0}{\int_{1/3}^{2/3}(0)(0)dx} + \cancelto{0}{\int_{2/3}^{1}(0)(3)dx} = 0$$

$$[K_{21}] = \int_0^1 \frac{d}{dx}N_2 \frac{d}{dx}N_1 dx = \int_0^{1/3}(3)(-3)dx + \cancelto{0}{\int_{1/3}^{2/3}(-3)(0)dx} + \cancelto{0}{\int_{2/3}^{1}(0)(0)dx} = -9x\Big|_0^{1/3} = (-9)\left[\frac{1}{3} - 0\right] = -3$$

$$[K_{22}] = \int_0^1 \frac{d}{dx}N_2 \frac{d}{dx}N_2 dx = \int_0^{1/3}(3)(3)dx + \int_{1/3}^{2/3}(-3)(-3)dx + \cancelto{0}{\int_{2/3}^{1}(0)(0)dx}$$

$$= 9x\Big|_0^{1/3} + +9x\Big|_{1/3}^{2/3} = 9\left[\frac{1}{3} - 0\right] + 9\left[\frac{2}{3} - \frac{1}{3}\right] = 3 + 3 = 6$$

$$[K_{23}] = \int_0^1 \frac{d}{dx}N_2 \frac{d}{dx}N_3 dx = \cancelto{0}{\int_0^{1/3}(-3)(0)dx} + \int_{1/3}^{2/3}(-3)(3)dx + \cancelto{0}{\int_{2/3}^{1}(0)(-3)dx} = -9x\Big|_{1/3}^{2/3} = -9\left[\frac{2}{3} - \frac{1}{3}\right] = -3$$

$$[K_{24}] = \int_0^1 \frac{d}{dx}N_2 \frac{d}{dx}N_4 dx = \cancelto{0}{\int_0^{1/3}(3)(0)dx} + \cancelto{0}{\int_{1/3}^{2/3}(-3)(0)dx} + \cancelto{0}{\int_{2/3}^{1}(0)(3)dx} = 0$$

$$[K_{31}] = \int_0^1 \frac{d}{dx}N_3 \frac{d}{dx}N_1 dx = \int_0^{1/3} \cancelto{0}{(0)(-3)}dx + \int_{1/3}^{2/3} \cancelto{0}{(3)(0)}dx + \int_{2/3}^{1} \cancelto{0}{(-3)(0)}dx = 0$$

$$[K_{32}] = \int_0^1 \frac{d}{dx}N_3 \frac{d}{dx}N_2 dx = \int_0^{1/3} \cancelto{0}{(0)(3)}dx + \int_{1/3}^{2/3} (3)(-3)dx + \int_{2/3}^{1} \cancelto{0}{(-3)(0)}dx = -9x\Big|_{1/3}^{2/3} = -9\left[\frac{2}{3} - \frac{1}{3}\right] = -3$$

$$[K_{33}] = \int_0^1 \frac{d}{dx}N_3 \frac{d}{dx}N_3 dx = \int_0^{1/3} \cancelto{0}{(0)(0)}dx + \int_{1/3}^{2/3} (3)(3)dx + \int_{2/3}^{1} (-3)(-3)dx$$

$$= 9x\Big|_{1/3}^{2/3} + 9x\Big|_{2/3}^{1} = 9\left[\frac{2}{3} - \frac{1}{3}\right] + 9\left[1 - \frac{2}{3}\right] = 3 + 3 = 6$$

$$[K_{34}] = \int_0^1 \frac{d}{dx}N_3 \frac{d}{dx}N_4 dx = \int_0^{1/3} \cancelto{0}{(0)(0)}dx + \int_{1/3}^{2/3} \cancelto{0}{(3)(0)}dx + \int_{2/3}^{1} (-3)(3)dx = -9x\Big|_{2/3}^{1} = -9\left[1 - \frac{2}{3}\right] = -3$$

$$[K_{41}] = \int_0^1 \frac{d}{dx}N_4 \frac{d}{dx}N_1 dx = \int_0^{1/3} \cancelto{0}{(0)(-3)}dx + \int_{1/3}^{2/3} \cancelto{0}{(0)(0)}dx + \int_{2/3}^{1} \cancelto{0}{(3)(0)}dx = 0$$

$$[K_{42}] = \int_0^1 \frac{d}{dx}N_4 \frac{d}{dx}N_2 dx = \int_0^{1/3} \cancelto{0}{(0)(3)}dx + \int_{1/3}^{2/3} \cancelto{0}{(0)(-3)}dx + \int_{2/3}^{1} \cancelto{0}{(3)(0)}dx = 0$$

$$[K_{43}] = \int_0^1 \frac{d}{dx}N_4 \frac{d}{dx}N_3 dx = \int_0^{1/3} \cancelto{0}{(0)(0)}dx + \int_{1/3}^{2/3} \cancelto{0}{(0)(3)}dx + \int_{2/3}^{1} (3)(-3)dx = -9x\Big|_{2/3}^{1} = -9\left[1 - \frac{2}{3}\right] = -3$$

$$[K_{44}] = \int_0^1 \frac{d}{dx}N_4 \frac{d}{dx}N_4 dx = \int_0^{2/3} \cancelto{0}{(0)(0)}dx + \int_{2/3}^{1} (3)(3)dx = 9x\Big|_{2/3}^{1} = 9\left[1 - \frac{2}{3}\right] = 3$$

Therefore, the stiffness matrix is,

$$[K_{AB}] = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \quad \text{...Eq.07-4}$$

FORCE VECTOR

Force vector for this PDE problem is given by Eq.05-7.

$$\bar{F}_A = \{F_A\} = \int_0^1 N_A \mathcal{f} dx + N_A(0)\mathcal{h} - \int_0^1 \frac{d}{dx}N_A \frac{d}{dx}N_{n+1}\mathcal{g} dx$$

In this problem,

$$\mathcal{f} = x$$

$$u(1) = \mathcal{g} = 0$$

$$-\frac{d}{dx}u(0) = \mathcal{h} = 0$$

By the way, $N_A(0)$ is zero for all F_A except for A=1. Also, by definition, N_{n+1} is zero for all F_A except for A = n+1. Therefore, these terms $N_A(0)\mathcal{h}$ and $-\int_0^1 \frac{d}{dx}N_A \frac{d}{dx}N_{n+1}\mathcal{g} dx$ are applicable for the boundary conditions.

Therefore, the force vector can be simplified as:

$$\{F_A\} = \int_0^1 N_A \mathcal{f} dx + N_A\cancelto{0}{(0)}\mathcal{h} - \int_0^1 \frac{d}{dx}N_A \cancelto{0}{\frac{d}{dx}}N_{n+1}(1)\mathcal{g} dx = \int_0^1 N_A x dx$$

Now, let's solve for each component of the force vector.

$$F_1 = \int_0^1 N_1 x dx = \int_0^{1/3} (-3x+1)x dx + \int_{1/3}^1 \cancelto{0}{(0)} x dx = \int_0^{1/3} \left(-3x^2 + x\right) dx$$

$$= \left(-x^3 + \frac{1}{2}x^2\right)\Bigg|_0^{1/3} = -\left(\frac{1}{3}\right)^3 + \frac{1}{2}\left(\frac{1}{3}\right)^2 = -\frac{1}{27} + \frac{1}{18} = \frac{1}{54}$$

$$F_2 = \int_0^1 N_2 x dx = \int_0^{1/3} (3x) x dx + \int_{1/3}^{2/3} (-3x+2) x dx + \int_{2/3}^1 \cancelto{0}{(0)} x dx = \int_0^{1/3} \left(3x^2\right) dx + \int_{1/3}^{2/3} \left(-3x^2 + 2x\right) dx$$

$$= x^3\Big|_0^{1/3} + \left(-x^3 + x^2\right)\Big|_{1/3}^{2/3} = \left(\frac{1}{3}\right)^3 + \left[-\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2\right] - \left[-\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2\right]$$

$$= \frac{1}{27} + \left(-\frac{8}{27} + \frac{4}{9}\right) - \left(-\frac{1}{27} + \frac{1}{9}\right) = \frac{1}{9}$$

$$F_3 = \int_0^1 N_3 x dx = \int_0^{1/3} \cancelto{0}{(0)} x dx + \int_{1/3}^{2/3} (3x-1) x dx + \int_{2/3}^1 (-3x+3) x dx$$

$$= \int_{1/3}^{2/3} \left(3x^2 - x\right) dx + \int_{2/3}^1 \left(-3x^2 + 3x\right) dx = \left(x^3 - \frac{1}{2}x^2\right)\Bigg|_{1/3}^{2/3} + \left(-x^3 + \frac{3}{2}x^2\right)\Bigg|_{2/3}^1$$

$$= \left[\left(\frac{2}{3}\right)^3 - \frac{1}{2}\left(\frac{2}{3}\right)^2\right] - \left[\left(\frac{1}{3}\right)^3 - \frac{1}{2}\left(\frac{1}{3}\right)^2\right] + \left[-(1)^3 + \frac{3}{2}(1)^2\right] - \left[-\left(\frac{2}{3}\right)^3 + \frac{3}{2}\left(\frac{2}{3}\right)^2\right]$$

$$= \left[\frac{8}{27} - \frac{4}{18}\right] - \left[\frac{1}{27} - \frac{1}{18}\right] + \left[-1 + \frac{3}{2}\right] - \left[-\frac{8}{27} + \frac{12}{18}\right] = \frac{2}{9}$$

$$F_4 = \int_0^1 N_4 x dx = \int_0^{2/3} \overbrace{(0)x dx}^{0} + \int_{2/3}^{1} (3x-2)x dx = \int_{2/3}^{1} \left(3x^2 - 2x\right) dx$$

$$= \left(x^3 - x^2\right)\Big|_{2/3}^{1} = \left[(1)^3 - (1)^2\right] - \left[\left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2\right] = -\left[\frac{8}{27} - \frac{4}{9}\right] = \frac{4}{27}$$

Therefore, the force vector is,

$$\{F_A\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \\ 4/27 \end{Bmatrix} \quad \text{..........Eq.07-5}$$

MATRIX FORM

The matrix form of general FEM problem, from Eq.05-6,

$$\bar{F}_A = \tilde{K}_{AB} \bar{d}_B$$

Or

$$\{F_A\} = [K_{AB}]\{d_B\}$$

Therefore, inserting Eq.07-4 and Eq.07-5, the matrix form of this problem becomes,

$$\begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \\ 4/27 \end{Bmatrix} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix} \quad \text{..........Eq.07-6}$$

By the way, because of the boundary condition $u(1) = g = 0$, we know $d_4 = 0$.

Therefore, Eq.07-6 can be shown as:

$$\begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \\ 4/27 \end{Bmatrix} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \end{Bmatrix}$$

Reduce the matrix size:

$$\begin{Bmatrix} \boxed{\begin{matrix} 1/54 \\ 1/9 \\ 2/9 \end{matrix}} \\ \cancel{4/27} \end{Bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{matrix}} & \begin{matrix} \cancel{0} \\ \cancel{0} \\ \cancel{-3} \end{matrix} \\ \begin{matrix} \cancel{0} & \cancel{0} & \cancel{-3} \end{matrix} & \cancel{3} \end{bmatrix} \begin{Bmatrix} \boxed{\begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}} \\ \cancel{0} \end{Bmatrix}$$

Thus, we only need to solve for d_1, d_2, and d_3, and the matrix equation now becomes:

$$\begin{Bmatrix}1/54\\ 1/9\\ 2/9\end{Bmatrix} = \begin{bmatrix}3 & -3 & 0\\ -3 & 6 & -3\\ 0 & -3 & 6\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix} \quad \text{..Eq.07-7}$$

SOLVE MATRIX FORM BY GAUSS-JORDAN ELIMINATION

To solve Eq.07-7, you need to find an inverse of stiffness matrix. In general, that is,

$$\{d_B\} = [K_{AB}]^{-1}\{F_A\}$$

A review of linear algebra: $[K_{AB}]^{-1}[K_{AB}] = [I]$

One of the common methods to find an inverse of the matrix is Gauss-Jordan elimination method.

$$\left[\begin{array}{ccc|ccc}3 & -3 & 0 & 1 & 0 & 0\\ -3 & 6 & -3 & 0 & 1 & 0\\ 0 & -3 & 6 & 0 & 0 & 1\end{array}\right]$$

Row 2 $\leftarrow$ Row 2 + Row 1

$$\Rightarrow \left[\begin{array}{ccc|ccc}3 & -3 & 0 & 1 & 0 & 0\\ 0 & 3 & -3 & 1 & 1 & 0\\ 0 & -3 & 6 & 0 & 0 & 1\end{array}\right]$$

Row 1 $\leftarrow$ Row 1 + (2) * Row 2

$$\Rightarrow \left[\begin{array}{ccc|ccc}3 & -3 & 0 & 1 & 0 & 0\\ 0 & 3/2 & 0 & 1 & 1 & 1/2\\ 0 & -3 & 6 & 0 & 0 & 1\end{array}\right]$$

Row 2 $\leftarrow$ Row 2 + $\frac{1}{2}$ * Row 3

$$\Rightarrow \left[\begin{array}{ccc|ccc}3 & 0 & 0 & 3 & 2 & 1\\ 0 & 3/2 & 0 & 1 & 1 & 1/2\\ 0 & -3 & 6 & 0 & 0 & 1\end{array}\right]$$

Row 3 $\leftarrow$ Row 3 + (2) * Row 2

$$\Rightarrow \left[\begin{array}{ccc|ccc}3 & 0 & 0 & 3 & 2 & 1\\ 0 & 3/2 & 0 & 1 & 1 & 1/2\\ 0 & 0 & 6 & 2 & 2 & 2\end{array}\right]$$

Row 1 $\leftarrow$ $\frac{1}{3}$ * Row 1

Row 2 $\leftarrow$ $\frac{2}{3}$ * Row 2

Row 3 $\leftarrow$ $\frac{1}{6}$ * Row 3

$$\Rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 2/3 & 1/3\\ 0 & 1 & 0 & 2/3 & 2/3 & 1/3\\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3\end{array}\right]$$

Therefore, the inverse of the stiffness matrix is,

$$[K_{AB}]^{-1} = \begin{bmatrix}1 & 2/3 & 1/3\\ 2/3 & 2/3 & 1/3\\ 1/3 & 1/3 & 1/3\end{bmatrix}$$

Double check the inverse matrix by multiplying the inverse matrix by the original matrix to see if you can get identity matrix.

$$[K_{AB}]^{-1}[K_{AB}] = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix}$$

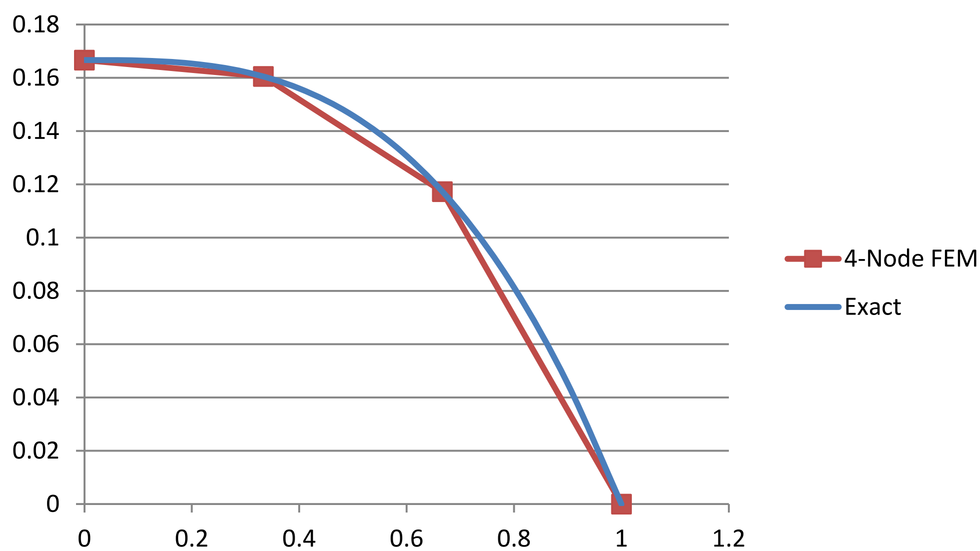
$$= \begin{bmatrix} 1\cdot(3)+\frac{2}{3}\cdot(-3)+\frac{1}{3}\cdot(0) & 1\cdot(-3)+\frac{2}{3}\cdot(6)+\frac{1}{3}\cdot(-3) & 1\cdot(0)+\frac{2}{3}\cdot(-3)+\frac{1}{3}\cdot(6) \\ \frac{2}{3}\cdot(3)+\frac{2}{3}\cdot(-3)+\frac{1}{3}\cdot(0) & \frac{2}{3}\cdot(-3)+\frac{2}{3}\cdot(6)+\frac{1}{3}\cdot(-3) & \frac{2}{3}\cdot(0)+\frac{2}{3}\cdot(-3)+\frac{1}{3}\cdot(6) \\ \frac{1}{3}\cdot(3)+\frac{1}{3}\cdot(-3)+\frac{1}{3}\cdot(0) & \frac{1}{3}\cdot(-3)+\frac{1}{3}\cdot(6)+\frac{1}{3}\cdot(-3) & \frac{1}{3}\cdot(0)+\frac{1}{3}\cdot(-3)+\frac{1}{3}\cdot(6) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ldots\text{OK}$$

Therefore,

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \end{Bmatrix} = \begin{Bmatrix} 1\cdot\frac{1}{54}+\frac{2}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{2}{9} \\ \frac{2}{3}\cdot\frac{1}{54}+\frac{2}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{2}{9} \\ \frac{1}{3}\cdot\frac{1}{54}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{3}\cdot\frac{2}{9} \end{Bmatrix} = \begin{Bmatrix} 1/6 \\ 13/81 \\ 19/162 \end{Bmatrix} = \begin{Bmatrix} 0.1667 \\ 0.1605 \\ 0.1173 \end{Bmatrix}$$

Finally we successfully solved for the displacement vectors. This concludes the first problem solving FEM by hand. The figure below shows a comparison of FEM approximation to the exact solution.



SOLVE MATRIX FORM BY EXCEL SPREADSHEET

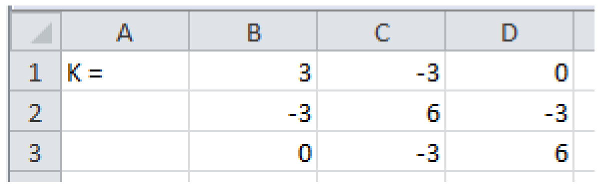
As we went over the problem, I believe everybody agreed that solving the matrix form is tedious, especially finding inverse of the stiffness matrix. Because solving the matrix is not the main topic of understanding

FEM theory, I want to use Excel spreadsheet. For these of who are not familiar to matrix operation in Excel, let's spend a few minutes here.

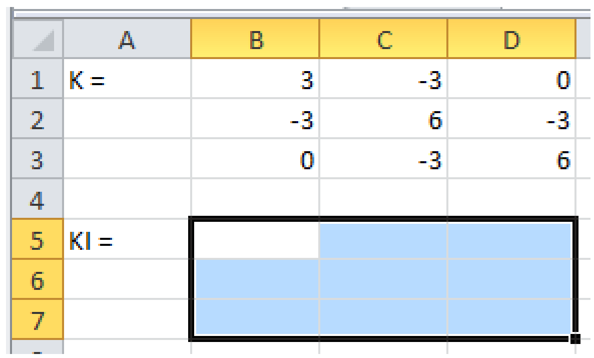
Let's solve the same matrix form but using Excel this time. The equation (Eq07-7) was,

$$\begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \end{Bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}$$

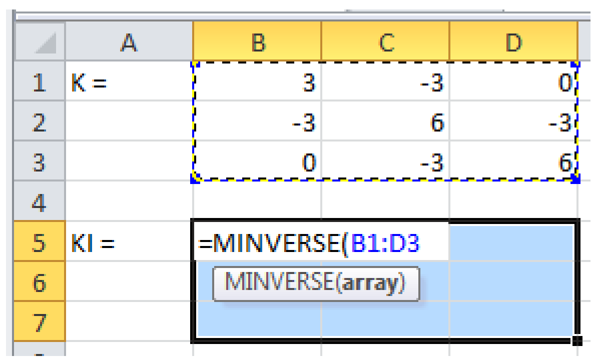
First, you need to enter the stiffness matrix as shown below (Cells B1 through D3).



Then, I want to compute the inverse of the matrix. First, you need to select the target cells. Because the original matrix is 3 by 3, the target cells should also be 3 x 3. In this example, cells B5 through D7 are selected.



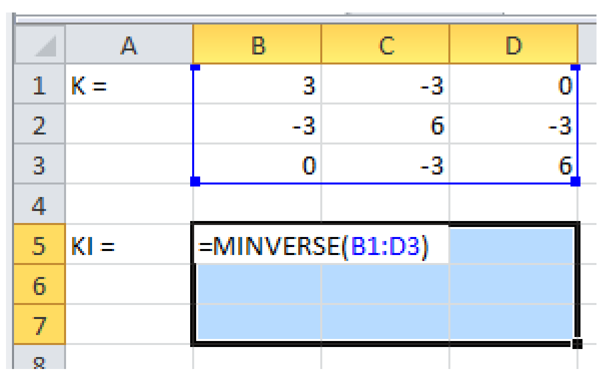
While selecting the cells, type **=MINVERSE(** and select the cells by dragging from B1 through D3.



Close the bracket but do not hit Enter key yet. The cells now look like this.

Then, while pressing **Ctrl key** and **Shift key**, press **Enter key**. (That is, **Ctrl + Shift + Enter**.)

Because of Ctrl + Shift + Enter,
someone may call it "**CSE formula**".



The inverse values of the matrix is now shown.

5	KI =	1	0.666667	0.333333
6		0.666667	0.666667	0.333333
7		0.333333	0.333333	0.333333

Note that the formula is shown with bracket "{". This is called "Array Formula" because it works with array data type.

fx {=MINVERSE(B1:D3)}

To check if the inverse value is correct, you can multiply the inverse matrix and original matrix to see if you can get identify matrix.

The formula to multiply matrix is **=MMULT(array1, array2)**.

1. Select **3x3** the target cells (B9:D11 in this example on the right figure).
2. Type **=MMULT(**
3. Select the first array (Inverse Matrix)
4. Type **,** (comma)
5. Select the second array (Original Matrix)
6. Type **)** to close bracket
7. Press **Ctrl + Shift + Enter**

	A	B	C	D
1	K =	3	-3	0
2		-3	6	-3
3		0	-3	6
4				
5	KI =	1	0.666667	0.333333
6		0.666667	0.666667	0.333333
7		0.333333	0.333333	0.333333
8				
9	KI*K =	=MMULT(B5:D7,B1:D3)		
10				
11				

Confirm that you get an identity matrix.

9	KI*K =	1	0	0
10		0	1	0
11		0	0	1

Now, let's define a force vector (F). In this example, it will be 1 by 3 matrix as shown in the figure right.

13	F =	0.018519
14		0.111111
15		0.222222

Displacement vector (d) is a multiplication of inverse of stiffness matrix (KI) and the force vector (F).

Again, use a matrix multiplication formula **=MMULT(array1, array2)**.

1. Select the **1x3** target cells (C17:C19 in this example on the right figure).
2. Type **=MMULT(**
3. Select the first array (Inverse Matrix)
4. Type **,** (comma)
5. Select the second array (Force Vector)
6. Type **)** to close bracket
7. Press **Ctrl + Shift + Enter**

	A	B	C	D
1	K =	3	-3	0
2		-3	6	-3
3		0	-3	6
4				
5	KI =	1	0.666667	0.333333
6		0.666667	0.666667	0.333333
7		0.333333	0.333333	0.333333
8				
9	KI*K =	1	0	0
10		0	1	0
11		0	0	1
12				
13	F =	0.018519		
14		0.111111		
15		0.222222		
16				
17	d = KI * F =		=MMULT(B5:D7,B13:B15)	
18				
19				
20				

Displacement vector values are now shown. This is the same values as what we got by hand calculation (of course).

17	d = KI * F =		0.166667	
18			0.160494	
19			0.117284	

The excel file containing the above examples can be download from the following link.

http:www.3dwhiffletree.com/FEM/Download/Chapter07.xlsx

UNDER CONSTRAINED PROBLEM – WHAT IF d_4 IS UNKNOWN?

The original stiffness matrix has 4 by 4 as we got in Eq.07-6. Then, we reduced it to 3 x 3 by eliminating the already-known displacement vector (d_4). In other words, can we solve this equation (Eq.07-6)?

$$\begin{Bmatrix} 1/54 \\ 1/9 \\ 2/9 \\ 4/27 \end{Bmatrix} = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

To solve for displacement vectors, you need to find the inverse of stiffness matrix.

In the same way, enter the original 4 x 4 matrix in the cells B1 through E4.

Select the target cells B6 through E4 and use **=MINVERSE(array)** formula and press **Ctrl + Shift + Enter**.

	A	B	C	D	E
1	K=	3	-3	0	0
2		-3	6	-3	0
3		0	-3	6	-3
4		0	0	-3	3
5					
6	KI=	=MINVERSE(B1:E4)			
7					
8					
9					
10					

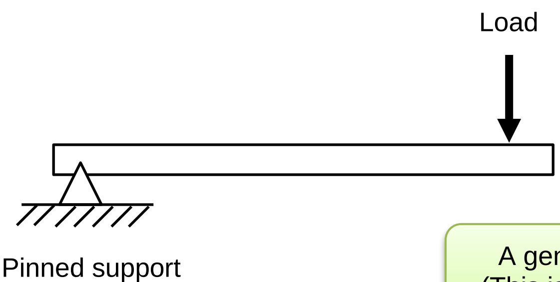
This time the solution for the inverse is failed.

6	KI=	#NUM!	#NUM!	#NUM!	#NUM!
7		#NUM!	#NUM!	#NUM!	#NUM!
8		#NUM!	#NUM!	#NUM!	#NUM!
9		#NUM!	#NUM!	#NUM!	#NUM!
10		#NUM!	#NUM!	#NUM!	#NUM!

There is a problem with a number used in the formula.

This is because the original equation has too many unknowns. In structural engineering problem, this is called the problem is **under constrained**. In such a case the stiffness matrix causes a **singularity** and the matrix form cannot be solved. The reason you get under-constrained could vary. You may have forgotten the boundary condition or the problem could be physically impossible to be stabilized.

An example of under-constrained problem is like this. This example is that the problem is not statically stable because there is only one pinned support and nothing else. The beam cannot resist against the moment caused by the load.



A general example of under-constrained
(This is not the above problem, by the way)